DAIR
DIAS Access to Institutional Repository

DIAS
Institiúid Ard-Léinn Bhaile Átha Cliath | Dublin Institute for Advanced Studies

Title	Lecture notes on supersymmetry.
Creators	O'Raifeartaigh, L.
Date	1975
Citation	O'Raifeartaigh, L. (1975) Lecture notes on supersymmetry. Communications of the Dublin Institute for Advanced Studies. ISSN Series A (Theoretical Physics) 0070-7414
URL	https://dair.dias.ie/id/eprint/129/

Sgríbhinní Institiúid Árd-Léinn Bhaile Átha Cliath
Sraith. A, Uimh. 22

Communications of the Dublin Institute for Advanced Studies
Series A (Theoretical Physics), No. 22

LECTURE NOTES ON SUPERSYMMETRY

by

L. O'RAIFEARTAIGH

DUBLIN
Institiúid Árd-Léinn Bhaile Átha Cliath
Dublin Institute for Advanced Studies
1975

Price: £2.10

Table of Contents

Introduction

Supersymmetry is a new symmetry that has emerged in recent years. It has the property that it allows fields of different spin, and, in particular, boson and fermion fields, to appear in the same irreducible multiplet.

The basic idea of supersymmetry was first put forward by Ramond[(1)] and later developed by Gervais and Sakita, Neveu and Schwarz, Iwasaki and Kikkawa[(2)] in the context of the dual model theory of scattering. Roughly speaking, the idea is to consider the 'square-root' of the Dirac operator in much the same way that Dirac considered the 'square-root' of the Klein-Gordon operator. More recently, supersymmetry was transferred by Wess and Zumino[(3),(4),(5)] from the context of the dual model, in which it operated in a 1 + 1 dimension to the context of conventional field theory in 3 + 1 dimensions. Later a more compact formulation in 3 + 1 dimensions, in terms of superfields, was introduced by Salam and Strathdee[(6)].

Supersymmetry has many remarkable properties. For example, in the renormalization of a supersymmetric Lagrangian all the quadratic divergences disappear and the number of independent renormalization constants is a minimum. Supersymmetry also allows a fully relativistic non-trivial mixing of Lorentz invariance and internal symmetry. It also allows the introduction of local (Yang-Mills) internal symmetry and improves the asymptotic freedom of conventional Yang-Mills theory. Finally, although supersymmetry is resistant to spontaneous symmetry breaking itself, it plays a role as catalyst for the spontaneous breakdown of internal symmetry.

The price that has to be paid for properties such as those just mentioned is, however, quite high. The difficulty is that, in their present simple form at any rate, supersymmetric multiplets are not observed in nature. The main reason for this is that all the fields in a given supermultiplet, including both fermions and bosons, must have the same mass and the same particle number, and such groups of fermions and bosons have not been seen. (Having the same particle number means that either the bosons have a conserved fermion number or the fermion number is conserved only modulo two.) It is true that those two problems have been overcome in certain very special models, but nevertheless their general resolution remains the outstanding problem of supersymmetry.

The present notes on supersymmetry developed out of some lectures given at the 1974 Aspen Center for Physics and attempt to give a short and simple survey of the principles and early developments of the theory. Although the notes can do no justice to the originality and power of the original papers on the subject quoted at the end, they have perhaps the secondary virtue of collecting the results of these papers together and presenting them in a form which, it is hoped, is easily readable without being too superficial.

By and large, the conventional development of the theory is followed here, but there are a few new features. These include the introduction of weight diagrams for the purpose of easy visualization, the explicit extraction of some useful sub-symmetries of supersymmetry, and some generalizations of earlier results on mass-breaking and on supersymmetric unified gauge theory. The range of topics covered in the notes is perhaps best summarized by the list of section headings given in the list of contents.

The author should like to take this opportunity to express his deep gratitude to the Directors of the Aspen Center for Physics and the Organizers of the Mathematical Physics Group for their kind invitation to visit the Center and for generous financial support*. In particular, he should like to thank Professor Arthur Jaffe for his continual encouragement, and many of the visitors at the centre, particularly J. Charap, D. Fairlie, J.-L. Gervais, D. Politzer, H.-S. Tsao and B. de Wit for enlightening and stimulating discussions. Some of the work was carried out at the 1974 Strobl (Austria) Workshop on Weak Interactions and the author should like to thank the organizers of the Workshop for their hospitality, and to thank many of the participants, particularly A. Balachandran, Z. Horvath, J. Nilsson, A. Pais and H. Stremnitzer, for helpful discussions and comments. I would like also to acknowledge editorial assistance from E. R. Wills.

* Supported in part by the National Science Foundation, under Grant NSF GP 40354X

1. Supersymmetry and Superfields.

Supersymmetry is most conveniently described in terms of superfields. A superfield is defined to be a field $\Phi(x\theta)$ which depends not only on the space-time coordinates x but also on a set of real (Majorana) spinors θ which anti-commute,

$$\{\theta_\alpha, \theta_\beta\} = 0, \qquad \theta_\alpha^* = \theta_\alpha, \qquad \alpha = 1 \ldots 4. \tag{1.1}$$

The crucial assumption here is the anti-commutativity, which characterizes the supersymmetry. The reality condition is assumed in order to reduce the number of conventional fields contained in the superfield $\Phi(x\theta)$, as will be seen below. To make the reality condition compatible with Lorentz invariance, one uses a basis (Majorana basis) in which the Dirac representation of the Lorentz group becomes real. (The real representation, and a discussion of Majorana spinors which is independent of the basis (and metric), is given in appendix A.) Since there are only sixteen independent functions that one can construct with θ satisfying (1.1) each component of a superfield contains sixteen components of conventional fields. To exhibit the conventional fields explicitly, we expand $\Phi(x\theta)$ in θ to obtain

$$\Phi(x,\theta) = A + \theta\cdot\psi + \tfrac{1}{4}\,\theta\cdot\left[F + \gamma_5 G + i\gamma_5 \not{B}\right]\theta + \tfrac{1}{4}(\theta\cdot\theta)(\theta\cdot\chi) + \tfrac{1}{32}(\theta\cdot\theta)^2 D, \tag{1.2}$$

where $\xi\cdot\eta \equiv \bar{\xi}\eta = \tilde{\xi}C\eta$, C being the charge conjugation matrix (appendix A). If the superfield is a scalar, i.e. has no external Lorentz indices, and is assumed to have positive parity, then the fields A, F, D in (1.2) are scalars, ψ, χ are spinors, G is a pseudo-scalar and B_μ is a pseudo-vector.

In general, the Poincaré transformation properties of $\Phi(x\theta)$ are

$$\left(U(a,\Lambda)\Phi\right)(x,\theta) = \mathcal{D}(\Lambda)\,\Phi\left(\Lambda^{-1}(x-a), S^{-1}\theta\right), \qquad \left(P\Phi\right)(x\theta) = \eta\,\Phi\left(x^P, i\gamma_0\theta\right), \tag{1.3}$$

where η is the intrinsic parity. That is to say, they are the conventional ones, except that θ must also be transformed (according to the Dirac representation S). From (1.2) we see that the conventional fields (A,F,D) (ψ,χ) and (G,B_μ) contained in $\Phi(x,\theta)$ then have the intrinsic parities $\eta, i\eta, -\eta$ respectively. The factor i for γ_0 and for the intrinsic parity of the fermions is necessary on account of the Majorana condition[7].

What characterizes a superfield, however, is that it also satisfies a supersymmetric transformation law, namely,

$$(U(\varepsilon)\Phi)(x_\mu,\theta) = \Phi(x_\mu + \tfrac{i}{2}\theta\cdot\gamma_\mu\varepsilon,\ \theta+\varepsilon), \tag{1.4}$$

where the ε_α are Majorana spinors satisfying (1.1) and anti-commuting with the θ_α. Note that the supersymmetric transformations induce translations in coordinate space, though not conversely. This property is perhaps most explicitly exhibited by the infinitesimal generators of the supersymmetric transformations (1.4) which are clearly

$$G_\alpha = \frac{\partial}{\partial\theta_\alpha} + \frac{i}{2}(\theta C\gamma_\mu)_\alpha \partial_\mu . \tag{1.5}$$

Together with the infinitesimal generators of translations, these form the algebra[8]

$$\{G_\alpha, G_\beta\} = i(C\gamma_\mu)_{\alpha\beta}\partial_\mu , \tag{1.6}$$

$$[\partial_\mu, G_\alpha] = 0, \quad i\partial_\mu \equiv P_\mu . \tag{1.7}$$

Thus the G_α generate P_μ but P_μ commutes with the G_α. It is interesting to compare (1.5) with the Dirac equation

$$\{\not{P}, \not{P}\} = 2P^2 . \tag{1.8}$$

One observes that just as $\not{P}$ is the 'square-root' of the Klein-Gordon operator P^2, so G_α is a 'square-root' of the Dirac operator $C\not{P}$. Thus G_α is in a sense the 'fourth-root' of the Klein-Gordon operator.

In the above derivation of (1.6) we have assumed that θ and ε are independent of x, and one might ask what happens when this assumption is dropped. The answer is that if θ and ε are linear in x, then the transformation $x_\mu \to x_\mu + \frac{i}{2}\theta\cdot\gamma_\mu\varepsilon$ becomes bilinear and becomes, in fact, the whole conformal group instead of the translation group. This is actually the case that was considered in the first paper (3) on supersymmetry in four dimensions.

We have not assumed in the above that the superfield $\mathcal{D}(\Lambda)$ is real, and indeed that assumption would be incompatible with Lorentz invariance unless the representation $\mathcal{D}(\Lambda)$ were real. In the case that $\mathcal{D}(\Lambda)$ is real, however, (for example in the case of the vector, or Majorana-Dirac, or especially the scalar representation) we may impose the condition

$$\Phi^*(x\theta) = \Phi(x\theta). \tag{1.9}$$

Here it is understood that complex conjugation interchanges the order of anti-commuting elements, and one easily sees from (1.2) that (1.9) then imposes reality conditions on the conventional fields contained in $\Phi(x\theta)$ also. In particular, when (1.9) is satisfied, the fermion fields ψ, χ must be Majorana fields. This result already foreshadows one of the major problems of supersymmetry, namely that fermion number may be conserved only modulo two.(9) If, on the other hand, we do not impose the reality condition (1.9) then a fermion number is defined since ψ and χ are complex, but then since the bosons are not real, this fermion number is also a boson number, which is equally unsatisfactory. This difficulty concerning fermion number is surmountable in some complicated models, but its immediate occurrence at this early stage is somewhat disquieting, and exemplifies the problems that may ensue from putting bosons and fermions on the same footing.

An important subsymmetry of superfields which will be used extensively in the sequel is the following: since the coefficient of the field A in the expansion (1.2) is unity we have the trivial identity

$$\Phi(A+\delta,\psi,\dots) = \Phi(A,\psi,\dots) + \delta \tag{1.10}$$

and hence if $P(\Phi)$ is any polynomial in the superfield we have

$$P(\Phi(A+\delta)) = P(\Phi(A)+\delta) = P'(\Phi(A)), \tag{1.11}$$

where $P'(\bar{\Phi})$ is a polynomial of the same degree but with different coefficients. Thus a shift in the A-field changes a polynomial in the superfield into a new polynomial of the same degree.

2. Weight Diagrams for Scalar Superfields.

In this revue we shall find it convenient to represent the expansion (1.2) by means of a weight diagram. We shall confine ourselves for simplicity to the case of a scalar superfield, leaving the discussion of higher spins to appendix B. For the scalar superfield the weight diagram corresponding to the expansion (1.2) is

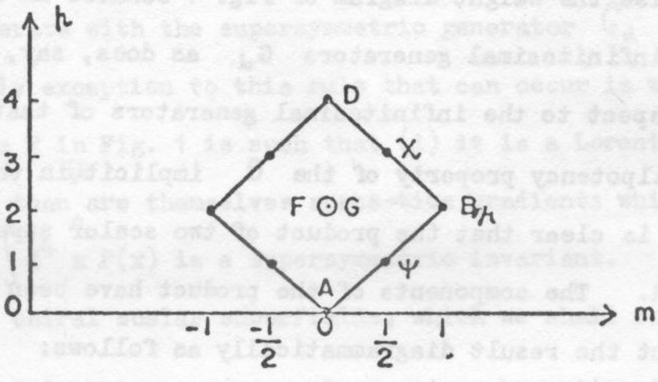


Fig. 1

where the vertical quantum number p = 0, 1, 2, 3, 4 represents the power of θ in (1.2) and the horizontal quantum number m = 0, $\pm\frac{1}{2}$, ± 1, ... represents the spin value in the representation of the Lorentz group to which the conventional field belongs. Note that spin multiplicities within irreducible representations of the Lorentz group are suppressed. Thus the four-component spinors ψ and χ show only two spin values $\pm\frac{1}{2}$, and the four-vector B_μ shows only three spin values 0, ± 1.

If we now return to the equations of (1.4), and let ε be infinitesimal, we find that the supersymmetric transformations take the form shown in Table 1,

$$
\begin{array}{lll}
\delta A & = \psi\cdot\delta\theta & \\
\delta\psi & = \frac{1}{2}(F+\gamma_5 G + i\gamma_5 \not{B})\delta\theta & -\frac{i}{2}(\not{\partial} A)\delta\theta \\
\delta F & = \frac{1}{2}\chi\cdot\delta\theta & -\frac{i}{2}\delta\theta\cdot\not{\partial}\psi \\
\delta G & = \frac{1}{2}\chi\cdot\gamma_5\delta\theta & -\frac{i}{2}\delta\theta\cdot\gamma_5\not{\partial}\psi \\
\delta B_\mu & = \frac{1}{2}\chi\cdot i\gamma_\mu\gamma_5\delta\theta & -\frac{i}{2}\delta\theta\cdot i\gamma_5\gamma_\mu\not{\partial}\psi \\
\delta\chi & = \frac{1}{2}D\delta\theta & -\frac{i}{2}(\not{\partial}F+\gamma_5\not{\partial}G+i\gamma_\nu\gamma_5\not{\partial})A_\nu \\
\delta D & & -i\delta\theta\cdot\not{\partial}\chi
\end{array}
$$

Table 1

where the G_α are the infinitesimal generators defined in (1.5). From these equations one sees by inspection that if one splits the generator G_α into two parts $\partial/\partial\theta_\alpha$ and $\frac{i}{2}(\theta\cdot\not\partial)_\alpha$, then $\frac{\partial}{\partial\theta}$ acts as a step-up operator for the conventional fields A, ψ, B_μ, and hence for the weights of the diagram of Fig. 1, while $\frac{i}{2}\,\theta\cdot\not\partial$ acts as a step-down operator. The only difference between the operation of $\partial/\partial\theta$ and $\frac{i}{2}\,\theta\cdot\not\partial$ and the operation of conventional ladder operators such as $L_\pm = L_1 \pm i L_2$ in SU(2) is that the step-down is accompanied by a space-time gradient $\not\partial$. Otherwise the weight diagram of Fig. 1 behaves in exactly the same way with respect to the infinitesimal generators G_α as does, say, the octet diagram of SU(3) with respect to the infinitesimal generators of that group.

Because of the nilpotency property of the θ implicit in the anti-commutation law (1.1) it is clear that the product of two scalar superfields is again a scalar superfield. The components of the product have been evaluated in refs.[3][10] and we may depict the result diagrammatically as follows:

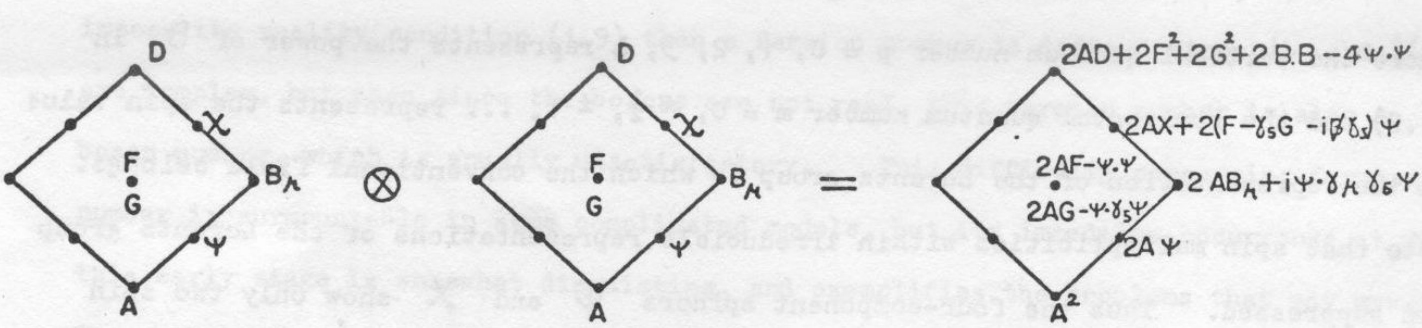


Fig. 2

This diagram will be very useful for reference in the sequel. Note that the contributions to the terms of weight p in the product come from terms in the components corresponding to the partitions of p into two ($p = p_1 + p_2$). If the component representations are not identical then the product is to be symmetrized.

3. Lagrangian Densities.

The fact that the product of any number of scalar superfields is again a scalar superfield and that the step-down operator is always accompanied by a space-time divergence means that already at this point we can say what the supersymmetric Lagrangians must be, namely the space-time integrals of the highest weights $D(x)$ in Figures such as Fig. 1. For if we step $D(x)$ up we obtain zero by definition, and if we step it down we obtain a space-time gradient, which vanishes on integration. Hence if we operate with the supersymmetric generator G_α on $\int d^4x\, D(x)$ we get zero. The only exception to this rule that can occur is when one of the lower weights such as F in Fig. 1 is such that (i) it is a Lorentz scalar and (ii) all the weights higher than it are themselves space-time gradients which vanish on integration. Then clearly $\int d^4 x\, F(x)$ is a supersymmetric invariant. This situation actually occurs for the chiral scalar superfields, which we shall consider in the next section.

We conclude this section on Lagrangians by noting that the subsymmetry (1.11) for polynomials in superfields can now be applied to Lagrangians. For the identity (1.11) holds even if the polynomial contains other superfields, and holds for each weight of the polynomial (which itself is a superfield) separately. Hence it holds for Lagrangians, which are just the highest weights of such polynomials. In other words, if $\mathcal{L}(A,\psi\ldots)$ is a supersymmetric Lagrangian density, then

$$\mathcal{L}(A+\delta,\psi\ldots) = \mathcal{L}'(A,\psi,\ldots) \tag{3.1}$$

where $\mathcal{L}'$ is a Lagrangian density of the same kind, but with different values of the coefficients. We shall see some specific instances of the subsymmetry (3.1) in the sequel.

4. Chiral Scalar Superfields.

It has been shown in refs. (3), (10) that the scalar superfields $\Phi(x\theta)$ of section 1 are not the smallest superfields that can be constructed. By defining the quantities

$$G^*_\alpha = \frac{\partial}{\partial\theta_\alpha} - \frac{i}{2}(\theta C \not\partial)_\alpha , \tag{4.1}$$

(i.e. similar to G_α but with a minus sign), and noting that they anti-commute with the G_α, one can define chiral scalar superfields $\Psi_\pm(x,\theta)$ as scalar superfields which satisfy the subsidiary conditions

$$\left[\tfrac{1}{2}(1\pm\gamma_5)G^*\right]_\alpha \Psi_\pm(x,\theta) = 0 , \tag{4.2}$$

and these turn out to be smaller than general scalar superfields. In fact, if one applies the conditions (4.2) to the expansion (1.2) one finds that $\Psi_\pm$ are of the form

Fig. 3

That is to say, only the fields A, ψ and F of the general superfield are independent. This means that we can adopt as a shorthand diagrammatic representation of a chiral superfield the small diagram

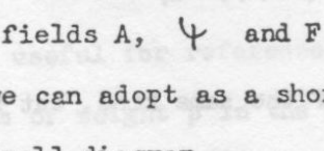


Fig. 4

corresponding to the small dotted diagram contained in Fig. 3. Note that the superfield consisting of the sum of two chiral superfields of opposite chirality, namely,

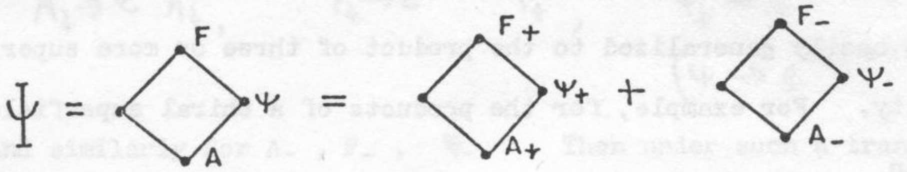
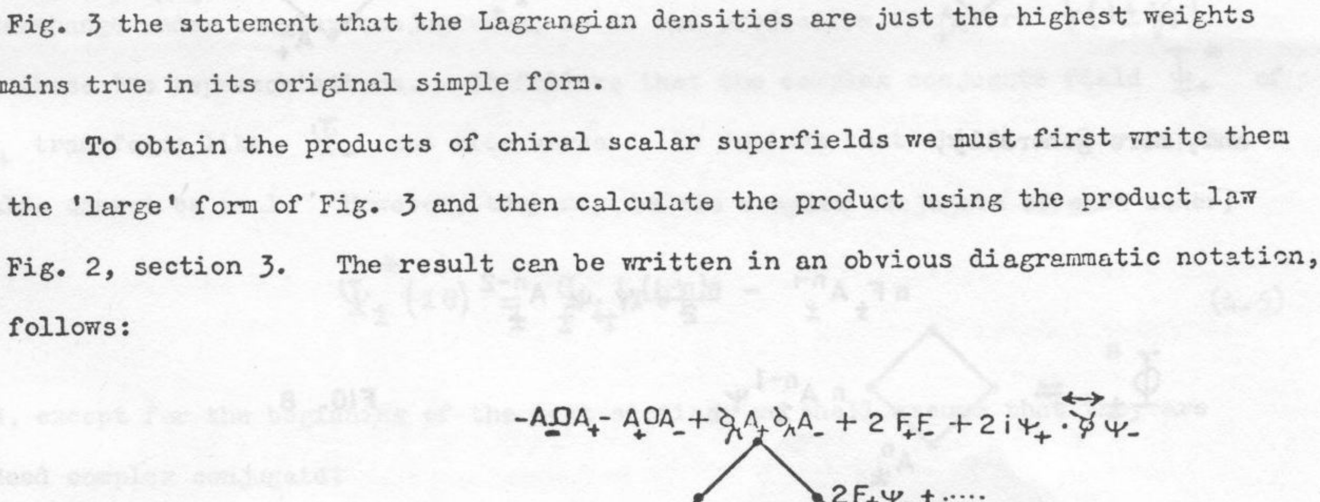


Fig. 5

is not a general chiral superfield. Note also that, as mentioned at the end of the last section, F(x) is a suitable candidate for a Lagrangian density. Thus if we use the smaller shorthand graphs of Fig. 4 and 5 in place of the full graphs such as Fig. 3 the statement that the Lagrangian densities are just the highest weights remains true in its original simple form.

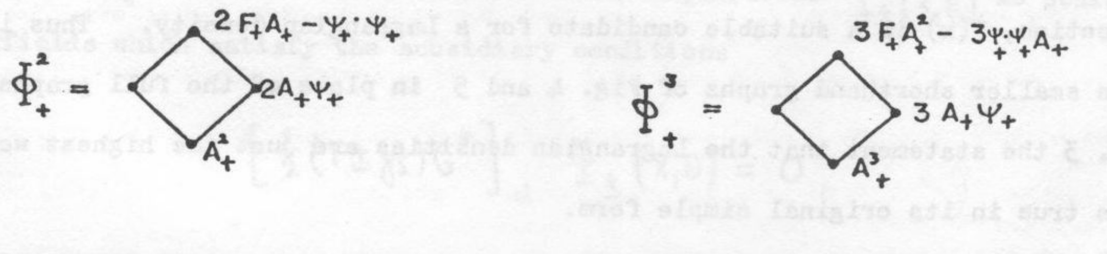
To obtain the products of chiral scalar superfields we must first write them in the 'large' form of Fig. 3 and then calculate the product using the product law of Fig. 2, section 3. The result can be written in an obvious diagrammatic notation, as follows:

FIG. 6



Thus the product of two scalar fields of opposite chirality is a 'large' superfield, but one in which the highest weight is closely related to the kinetic part of a Lagrangian density. The product of two superfields of the same chirality, on the other hand, is again a superfield of that chirality, and with no derivatives. The latter result is easily generalized to the product of three or more superfields of the same chirality. For example, for the products of a chiral superfield with itself, we obtain

FIG.7



and, more generally,

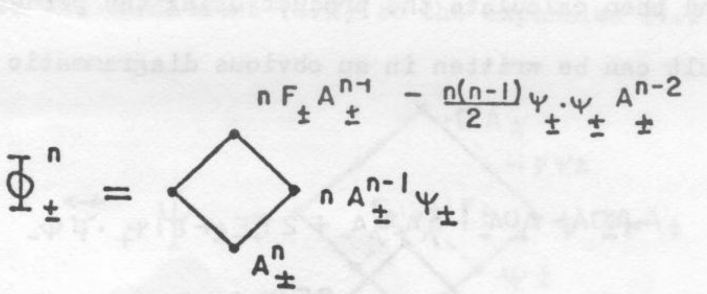


FIG. 8

This completes our discussion of the weight diagrams, except for higher spin which is discussed in appendix B. Before proceeding to use them to construct the Wess-Zumino Lagrangian, however, we wish to draw attention to two subsymmetries of supersymmetry which are very simple when expressed in terms of the diagrams and are of great help later in understanding the properties of the WZ-Lagrangian. The subsymmetries are

$$\text{I} \qquad \frac{\partial F_n^\pm}{\partial A_n^\pm} = n F_{n-1}^\pm \,. \qquad (4.3)$$

This subsymmetry follows from inspection of Fig. 11. It is clearly a special case of the identity (1.11) derived in section 1.

II Let C(m) denote elements of the family of chiral transformations*

$$A_+ \to e^{i\phi} A_+ , \qquad F_+ \to e^{-i(m-1)\phi} F_+ , \qquad \psi_+ \to e^{-\frac{i}{2}(m-2)\phi} \psi_+ \tag{4.4}$$
$$\left(\psi \to e^{-i/2 (m-2)\gamma_5 \phi}\, \psi\right),$$

and similarly for A_- , F_- , ψ_- . Then under such a transformation

(a) $\mathcal{D}(\bar{\Psi}_+ \bar{\Psi}_-)$ remains invariant.

(b) $F_n^+ \to e^{i(n-m)\phi} F_n^+$. That is to say, F_n^+ transforms covariantly, remaining invariant only for the chiral transformations with the same integer n

Finally we note that since the Lorentz representations $\mathcal{D}(\tfrac12 0)$ and $\mathcal{D}(0\tfrac12)$ interchange under complex conjugation, so do the projection operators $\frac{1}{2}(1 \pm \gamma_5)$ for these two representations. It follows that the complex conjugate field Ψ_+^* of Ψ_+ transforms like Ψ_- and vice versa. It follows that chiral scalar superfields cannot be real. However, they may be the complex conjugate of each other,

$$\Psi_\pm^*(x\theta) = \Psi_\mp(x\theta) \tag{4.5}$$

and, except for the beginning of the next section, we shall assume that they are indeed complex conjugate.

The projections $\frac{1}{2}(1 \pm \gamma_5)$ also interchange under parity, and hence a natural definition of the parity operator is

$$(P\Psi_\pm)(x,\theta) = \eta\, \Psi_\mp(x^P, i\gamma_0\theta), \tag{4.6}$$

where η is the intrinsic parity. The phase of η is actually fixed only up to a gauge transformation of the form $\Psi_\pm \to e^{\pm i\alpha}\Psi_\pm$, but, as discussed in Appendix D, the gauge can be fixed relative to the sign of the mass-operator.

---

* These transformations can be written more compactly as the superfield transformations

$$\Phi_+(x,\theta) \to e^{i\phi}\, \Phi_+(x, e^{-\frac{im}{2}\gamma_5\phi}\theta).$$

5. Derivation of the Wess-Zumino Lagrangian.

We have seen in section 2 that the supersymmetric invariant Lagrangian must be formed from the highest weights of the products of superfields. Let us now form a supersymmetric Lagrangian satisfying the following three constraints:

(i) It should contain only spin 0 and spin $\frac{1}{2}$ fields;

(ii) The kinetic terms should be conventional (linear for the spin $\frac{1}{2}$ fields and quadratic for the spin 0 fields);

(iii) It should be renormalizable.

The first condition limits us to chiral scalar fields, since all other fields have at least spin 1. The second condition, together with hermiticity, limits us to the highest weights of $\Phi^*_\pm \Phi_\pm$ (Fig. 6) for the kinetic term. The third condition limits us to Φ_+^n and Φ_-^n for $n \leq 3$ for the mass and interaction terms. Finally parity invariance restricts us to combinations such as $\Phi_+^2 + \Phi_-^2$. Thus under the conditions stated, the most general supersymmetric Lagrangian is

$$\mathcal{L} = \mathcal{L}_{\text{kinetic}} + \mathcal{L}_{\text{scale}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{interaction}}$$

$$= H\left\{-\frac{1}{8}\left[\Phi_+^*\Phi_+ + \Phi_-^*\Phi_-\right]\right\} + \frac{H}{2}\left\{\frac{\lambda}{\sqrt{2}}(\Phi_+ + \Phi_-) + \frac{m}{2}(\Phi_+^2 + \Phi_-^2) + \frac{g}{3\sqrt{2}}(\Phi_+^3 + \Phi_-^3) + \text{h.c.}\right\} \quad (5.1)$$

where H denotes highest weight. If we now assume that $\Phi_\pm$ are conjugate, this Lagrangian reduces to

$$\mathcal{L} = H\left\{-\frac{1}{4}\Phi_+\Phi_- + \frac{\lambda}{2}(\Phi_+ + \Phi_-) + \frac{m}{2}(\Phi_+^2 + \Phi_-^2) + \frac{g}{3\sqrt{2}}(\Phi_+^3 + \Phi_-^3)\right\}. \quad (5.2)$$

From Figs. 6 and 7 we then see by inspection that (5.2) is just

$$\mathcal{L} = \left\{-\partial_\mu A_+ \partial_\mu A_- - \frac{1}{2}\psi_+ \cdot \partial\!\!\!/ \psi_- - \frac{1}{2}\psi_- \cdot \partial\!\!\!/ \psi_- + F_+ F_-\right\} + \frac{\lambda}{\sqrt{2}}\left\{F_+ + F_-\right\}$$
$$+ m\left\{F_+ A_+ + F_- A_- - \frac{1}{2}\psi_+ \cdot \psi_+ - \frac{1}{2}\psi_- \cdot \psi_-\right\} + \frac{g}{\sqrt{2}}\left\{F_+ A_+^2 + F_- A_-^2 - \frac{1}{2}\psi_+ \cdot \psi_+ A_+ - \frac{1}{2}\psi_- \cdot \psi_- A_-\right\}. \quad (5.3)$$

Now defining $A_\pm = (A \pm iB)/\sqrt{2}$, $F_\pm = (F \mp iG)/\sqrt{2}$ this becomes

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}(F^2 + G^2) + \lambda F + m(FA + GB + i\bar{\psi}\psi)$$
$$+ g\left\{F(A^2 - B^2) + 2GAB - \bar{\psi}(A + i\gamma_5 B)\psi\right\}. \quad (5.4)$$

The Wess-Zumino Lagrangian is just this Lagrangian with the scale parameter λ set equal to zero. (11) If we use the equations of motion to eliminate the fields F and G which have no kinetic terms, we obtain the Lagrangian in more familiar form,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A)^2 - \frac{1}{2}(\partial_\mu B)^2 - \frac{i}{2}\bar{\psi}\not{\partial}\psi - \frac{m^2}{2}(A^2+B^2) - \frac{im}{2}\bar{\psi}\psi$$
$$- g\bar{\psi}(A+i\gamma_5 B)\psi - mgA(A^2+B^2) - \frac{g^2}{2}(A^2+B^2)^2. \qquad (5.5)$$

6. Subsymmetries of the Wess-Zumino Lagrangian.

The WZ-Lagrangian is clearly highly symmetric, but for the subsequent discussion it will be convenient to formulate its symmetry properties explicitly. This we do by extracting from the general supersymmetry three subsymmetries which are more concrete, and at the same time will be sufficient for the discussion.

The first subsymmetry is just the second equation (1.7) of the supersymmetric algebra

$$[P_\mu, G_\alpha] = 0 \qquad (6.1)$$

One immediate consequence of this subsymmetry is that the different particles in the same supermultiplet (which are connected by the G_α) all have the same mass. The next two subsymmetries are concerned with maintaining this mass-equality under interaction.

The second subsymmetry is the symmetry of $\mathcal{L}_{k.e.} + \mathcal{L}_g$ under the chiral transformations

$$A+iB \to e^{i\phi}(A+iB) \qquad F+iG \to e^{2i\phi}(F+iG), \qquad \psi \to e^{\frac{i}{2}\gamma_5\phi}\psi \qquad (6.2)$$

This chiral symmetry is broken by $\mathcal{L}_\lambda$ and $\mathcal{L}_m$ and we easily recognize it as the chiral symmetry $C(3)$ of section 4. (Note that the simpler symmetry $C(2)$, which here becomes ,

$$A+iB \to e^{i\phi}(A+iB)\ ,\quad F+iG \to e^{i\phi}(F+iG)\ ,\quad \psi\to\psi\ , \tag{6.3}$$

is preserved by $\mathcal{L}_{k.e.}+\mathcal{L}_m$ and broken by $\mathcal{L}_\lambda$ and $\mathcal{L}_g$.)

The last subsymmetry is obtained by first noting empirically from (5.4) that

$$\frac{\partial\mathcal{L}_g}{\partial A}=\frac{2g}{m}\mathcal{L}_m\ ,\qquad \frac{\partial\mathcal{L}_m}{\partial A}=\frac{m}{\lambda}\mathcal{L}_\lambda \tag{6.4}$$

which we easily recognize as the subsymmetry $\frac{\partial F_n^{\pm}}{\partial A^{\pm}}=nF_{n-1}^{\pm}$ of section 4. Since (6.4) holds only before the F and G fields are eliminated, that is to say, with the partial derivatives taken keeping F and G fixed, it turns out to be more convenient to express this subsymmetry as the identity

$$\mathcal{L}(A+\delta\,,\ m-2g\delta\,,\ \lambda-m\delta-g\delta^2)=\mathcal{L}(A,m,\lambda) \tag{6.5}$$

for any real parameter δ . This identity, which we shall call the <u>scaling</u> identity, for reasons which will become clear in the section on spontaneous symmetry breaking, holds both before and after the elimination of the F and G fields. Note that (6.5) is a specific example of the general identity (3.1) derived in section 3.

We note finally that if we define the potential of the WZ-Lagrangian to be

$$V(A,B)=-\frac{m^2}{2}(A^2+B^2)-mgA(A^2+B^2)-\frac{g^2}{2}(A^2+B^2)^2\ , \tag{6.6}$$

that is to say to be the WZ-Lagrangian stripped of its fermion and kinetic terms, then the scaling identity holds also for $V(A,B)$ alone

$$V(A+\delta,\ m-2g\delta,\ \lambda-m\delta-g\delta^2)\ =\ V(A,m,\lambda)\ . \tag{6.7}$$

This identity will be very useful for the discussion of spontaneous symmetry breaking.

7. The Renormalization Properties of the Wess Zumino Lagrangian.

At first sight the WZ-Lagrangian looks like an ordinary Yukawa-P(ϕ^4) type theory with certain constraints imposed on the masses and coupling constants so that only one mass and one coupling constant are independent. However it is much more than that. The point is that, since the constraints are dictated by an underlying symmetry group, the correlations are expected to hold not merely in the tree approximation but also after renormalization. In particular one should be able to renormalize the theory without introducing any more masses and coupling constants. Thus, since all the masses are equal because of the subsymmetry

$$[P_\mu, G_\alpha] = 0 \tag{7.1}$$

and the interaction is expected to maintain this equality, we expect that we should have the same mass renormalization

$$Z_{m_A} = Z_{m_B} = Z_{m_\psi} \tag{7.2}$$

and the same wave-function renormalization

$$Z_A = Z_B = Z_\psi \tag{7.3}$$

for all particles. Similarly, given (7.2) and (7.3), the maintenance of the scaling identity

$$\mathcal{L}(A+\delta,\ m-2g\delta,\ \lambda-m\delta-g\delta^2) = \mathcal{L}(A,m,\lambda) \tag{7.4}$$

which correlates the scale of A and m, leads us to expect that the mass and wave-function renormalizations will be correlated. Thus, if the supersymmetry is to be maintained after renormalization we expect strong conditions on the renormalization constants.

Wess and Zumino[11] have carried out one-loop calculations to check whether such strong conditions do indeed hold. They find that in fact even stronger results hold than might be expected from the above considerations. Their results may be summarized as follows:

For the Lagrangian with the dummy fields F and G not yet eliminated they find that

(α) There is no divergent mass or coupling constant renormalization other than that induced by the wave-function renormalizations;

(β) All the wave-function renormalizations are equal;

(γ) The common wave-function renormalization constant is logarithmically divergent.

For the Lagrangian with F and G eliminated, these results translate into the following:

(a) There is only one common mass renormalization and it is logarithmically divergent;

(b) There is only one common wave-function renormalization and it is equal to the mass renormalization;

(c) The two Yukawa vertex renormalizations are finite.

Note that result (a) implies that the quadratic and linear divergences which one might expect for the boson self-masses vanish. This is because supersymmetry forces the bosons to behave like fermions, which have only quadratically divergent self-masses. Thus we get the result we expected in eq. (7.2) above, and the only question is how the supersymmetry arranges for this result to emerge. The answer can be found by looking at one of the boson self-masses, the A-boson say, which is of the form shown in Fig. 9.

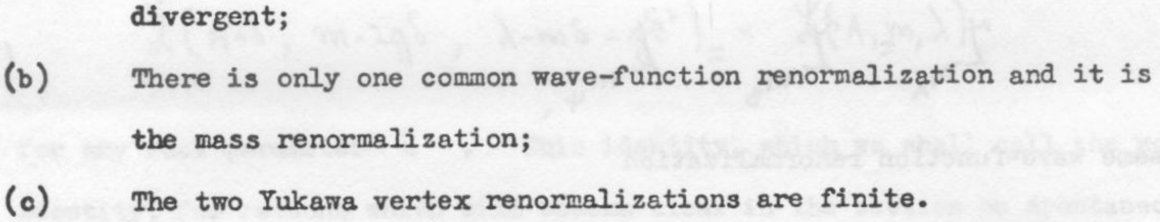


Fig. 9

Clearly the quadratic divergences cancel because the ϕ^4 and Yukawa coupling constants are the same, and this is so because the supersymmetry allows only one coupling constant altogether, and allows it in the particular way given by the WZ-Lagrangian. It is worth remarking in this connection that the result depends critically on the fact that the fermion field is a Majorana field.

The result (b) above shows that the expectation that the scaling identity would persist after renormalization is correct up to second order at least. The result (c) above comes from an entirely different source, however, namely from the chiral invariance of the WZ-Lagrangian mentioned in the last section. This can be seen by direct inspection of the third-order Yukawa vertices which one might expect to be logarithmically divergent (first two graphs of Fig. 10) but whose divergences actually cancel.

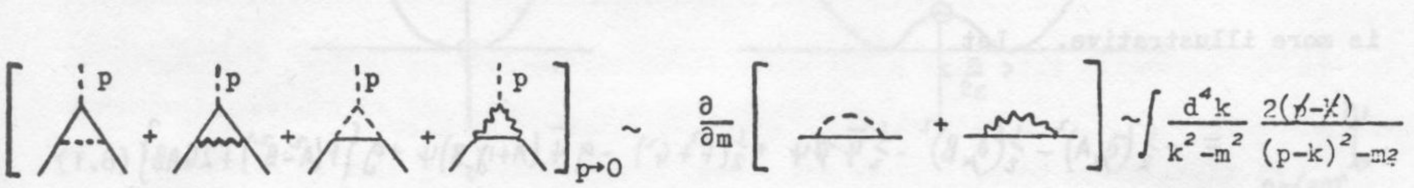


Fig. 10

The result can also be seen from the Ward identity for chiral invariance, the second order form of which, after the elimination of the F and G dummy fields, is shown in Fig. 10. (Note that since there is no mass in the numerator of the integrand on the right hand side the derivative with respect to mass makes the integral convergent.)
(12)(13)
The Ward identity has the advantage that it can be used to establish the convergence of the Yukawa couplings to all orders.

8. Spontaneous Symmetry Breaking and the WZ-Lagrangian.

In order to obtain more realistic models it would be desirable to introduce a splitting between the fermion and boson masses in supersymmetry. At the same time, one should like to split the masses in such a way that most of the 'good' properties of supersymmetry (e.g. absence of linear and quadratic divergences) would be retained. The obvious candidate for such splitting would be spontaneous symmetry breaking.[12][14] However, it is now known that the simple WZ-Lagrangian of section five cannot be spontaneously broken. In this section we wish to give a simple explanation for this result, namely, that the WZ-Lagrangian is already spontaneously broken. Thus it is already at a potential minimum and so (for a simple potential) cannot be shifted except to a maximum or to another supersymmetric minimum.

To show this we proceed in a somewhat indirect way, which however we hope is more illustrative. Let

$$\mathcal{L}_{m=\lambda=0} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}(F^2+G^2) - g\bar{\psi}(A+i\gamma_5 B)\psi + g\{F(A^2-B^2)+2GAB\} \qquad (8.1)$$

be the chirally invariant WZ-Lagrangian with mass and scale parameters zero. Now introduce a scale term $-\lambda F$ (where we can take the sign of λ negative without loss of generality because its sign can be changed by the chiral transformation (6.2) with $\phi = \pi/2$). Then eliminating the fields F and G in the usual way we obtain

$$\mathcal{L}_{m=0} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi - g\bar{\psi}(A+i\gamma_5 B)\psi - \frac{g^2}{2}(A^2+B^2)^2 + \lambda g(A^2-B^2). \qquad (8.2)$$

From this equation we see that the effect of the scale term λ is to break the chiral invariance by adding to the Lagrangian a term of the form $\lambda g(A^2-B^2)$. Furthermore we see that this term is tachyonic since it gives the field A^2 a negative mass^2 . However, before dismissing it for this reason we observe that the scale term and the previous chiral-invariant potential combine in the following interesting way

$$-\frac{g^2}{2}\left(A^2+B^2\right)^2 + \lambda g\left(A^2-B^2\right) = -\frac{g^2}{2}\left\{\left(A+\frac{m}{2g}\right)^2+B^2\right\}\left\{\left(A-\frac{m}{2g}\right)^2+B^2\right\}, \tag{8.3}$$

where $\qquad m = 2\sqrt{\lambda g}\ .$

What this equation shows is that the effect of the scale term is to change the potential of Figure 11a to that of Figure 11b, where for $\langle A\rangle = \langle B\rangle = 0$ we are at the encircled points.

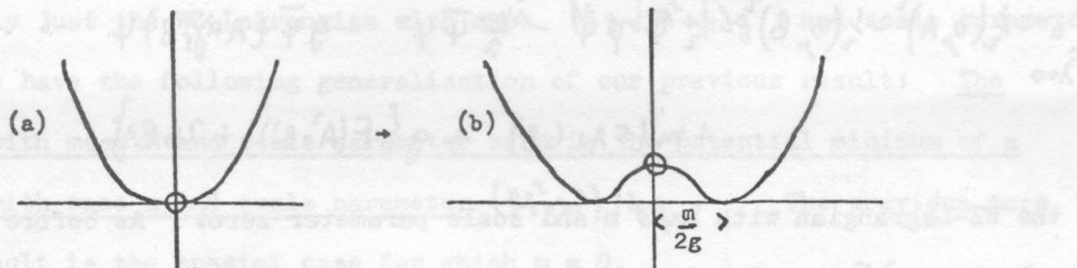


Fig. 11

Thus the scale term has provided a natural mechanism for introducing a Goldstone potential. The Goldstone potential respects the supersymmetry but breaks the chiral symmetry. If we now shift our fields to one of the minima of Fig. 11b by letting

$$A = a + \frac{m}{2g}, \qquad B = b, \qquad \langle a\rangle = \langle b\rangle = 0, \tag{8.4}$$

we obtain

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{2}(\partial_\mu a)^2 - \tfrac{1}{2}(\partial_\mu b)^2 - \tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{im}{2}\bar{\psi}\psi \\ &- g\bar{\psi}(a+i\gamma_5 b)\psi - \frac{g^2}{2}\left\{a^2+b^2\right\}\left\{\left(a+\frac{m}{g}\right)^2+b^2\right\}.\end{aligned} \tag{8.5}$$

But if we expand the potential term in this Lagrangian we see that it is just the massive WZ-Lagrangian of section 6 with $m = 2\sqrt{\lambda g}$. Thus we have the result: The WZ-Lagrangian with mass m and scale parameter zero, is the potential minimum of a WZ-Lagrangian with mass zero and scale parameter $m^2/4g$. Since there are only two minima in the potential of Fig. 11, and both are supersymmetric, this explains why the WZ-Lagrangian cannot be spontaneously broken. Note that this result also justifies our calling the term $-\lambda F$ a scale term, since the scale-parameter λ introduces a mass into the theory by the above mechanism.

Before discussing the deeper reason for the above result, it is better to generalize it as follows: Let

$$\mathcal{L}_{\lambda=0} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{i}{2}\bar{\psi}\not{\partial}\psi - \tfrac{im}{2}\bar{\psi}\psi - g\bar{\psi}(A+i\gamma_5 B)\psi$$
$$+ m(FA+GB) + g\left\{F(A^2-B^2) + 2GBA\right\} \tag{8.6}$$

be the WZ-Lagrangian with mass m and scale parameter zero. As before, introduce a scale term $-\lambda F$. On eliminating F and G we obtain this time

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{i}{2}\bar{\psi}\not{\partial}\psi - \tfrac{im}{2}\bar{\psi}\psi - g\bar{\psi}(A+i\gamma_5 B)\psi$$
$$-\frac{g^2}{2}\left\{A^2+B^2\right\}\left\{\left(A+\frac{m}{g}\right)^2+B^2\right\} + \lambda g\left\{\left(A+\frac{m}{2g}\right)^2 - B^2\right\}. \tag{8.7}$$

Thus in this case the effect of the scale term is to introduce a term $\lambda g\left\{\left(A+\frac{m}{2g}\right)^2 - B^2\right\}$. For $m \neq 0$ this term in linear in A and produces neither a potential maximum nor minimum. However once again we find that the scale term combines with the original potential in a simple way, namely,

$$-\frac{g^2}{2}\left\{A^2+B^2\right\}\left\{\left(A+\frac{m}{g}\right)^2+B^2\right\} + \lambda g\left\{\left(A+\frac{m}{2g}\right)^2 - B^2\right\} = -\frac{g^2}{2}\left\{\left(A+\frac{m+M}{2g}\right)^2+B^2\right\}\left\{\left(A+\frac{m-M}{2g}\right)^2+B^2\right\} \tag{8.8}$$

where $M^2 = m^2 + 4\lambda g$. Thus in the massive case the effect of the scale term is to keep the general form of the original potential unchanged, but to shift it from

Fig. 12a to Fig. 12b, where for $\langle A\rangle = \langle B\rangle = 0$ we are at the circled points. Hence in 12b we are not at a turning point of the potential but on the slope. If we now shift as before to the new potential minimum by letting

$$A = a + \frac{M-m}{2g}, \quad B = b, \quad \langle a\rangle = \langle b\rangle = 0, \tag{8.9}$$

then (8.7) becomes

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu a)^2 - \tfrac{1}{2}(\partial_\mu b)^2 - \tfrac{1}{2}\bar{\psi}(\not\partial + M)\psi - g\bar{\psi}(a - \gamma_5 b)\psi - \frac{g^2}{2}\left[a^2+b^2\right]\left[\left(a+\frac{M}{g}\right)^2 + b^2\right], \tag{8.10}$$

which is clearly just the WZ-Lagrangian with mass $M = \sqrt{m^2 + 4\lambda g}$ and scale parameter zero. Thus we have the following generalization of our previous result: <u>The WZ-Lagrangian with mass M and scale parameter zero is the potential minimum of a WZ-Lagrangian with mass m and scale parameter</u> $(M^2 - m^2)/4g$. The previous more interesting result is the special case for which m = 0.

Stated in this general form our result can be understood immediately in terms of the third symmetry of section 6, namely,

$$\mathcal{L}(A+\delta,\ m-2g\delta,\ \lambda - m\delta - g\delta^2) = \mathcal{L}(A, m, \lambda). \tag{8.11}$$

For from (8.7) we see that if we introduce a scale term $-\lambda F$ into the Lagrangian, it induces a term $\lambda g\left[\left(A+\frac{m}{2g}\right)^2 - B^2\right]$ which moves us away from the potential minimum (since if $m \neq 0$ it produces a term linear in A, and if m = 0 there are no other mass terms present and it introduces the tachyonic term $\lambda g A^2$ for A). Hence to return to a potential minimum we must eliminate λ. But from (8.11) we can do that by suitably shifting the field. Indeed if, as a special case of (8.11), we choose

$$\lambda = m\delta_0 + g\delta_0^2,$$

then we have

$$\mathcal{L}(a, M, 0) = \mathcal{L}(A, m, \lambda), \tag{8.12}$$

where

$$a = A + \delta_0, \qquad M = m - 2g\delta_0 = \sqrt{m^2 + 4\lambda g}, \tag{8.13}$$

which is just the result (8.10). Thus the scaling identity (8.11) provides a simple explanation of the results found empirically above.

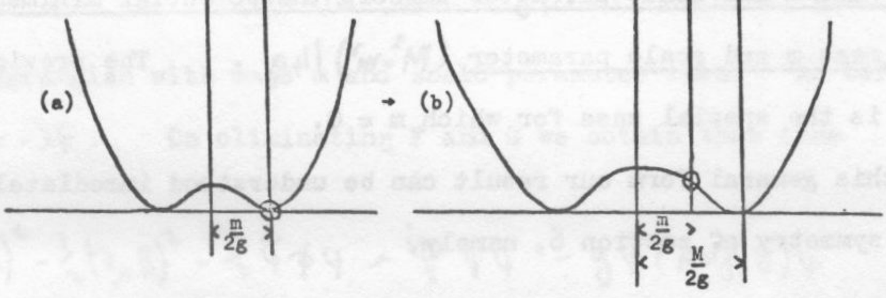


Fig. 12

9. Spontaneous Symmetry Breaking for more than one Superfield

We have seen in the last section that spontaneous symmetry breaking is not possible for the WZ-Lagrangian for one superfield. However, it has been shown by Fayet and Illiopoulos (FI) that this negative result is not necessarily a general feature of supersymmetry.[15] In fact, FI have constructed a counter-example using two chiral and one non-chiral superfields, and allowing parity to be violated after symmetry breaking. We shall not reproduce the FI model here, as a summary would not do the original version justice. Instead we shall investigate the spontaneous symmetry breaking situation for a system of n chiral scalar superfields. This will help put the results of the last section in perspective, and will be useful when we come to consider internal symmetry. The result of the investigation will be that, in general, spontaneous symmetry breakdown does <u>not</u> occur for chiral scalar superfields, but that it may occur in certain singular cases.

We consider, therefore, N chiral scalar superfields $\Psi^a_\pm$, $a=1\ldots N$ satisfying the conjugacy condition

$$\left(\Psi^a_\pm\right)^* = \left(\Psi^a_\mp\right). \tag{9.1}$$

The analogue of the WZ Lagrangian for such a system is

$$\mathcal{L} = \mathcal{L}_{k.e.} + \lambda_a \Psi^a_+ + m_{ab}\Psi^a_+\Psi^b_+ + g_{abc}\Psi^a_+\Psi^b_+\Psi^c_+ + \ldots \tag{9.2}$$

where the m_{ab}, g_{abc} are totally symmetric. In particular the fermion mass-matrix is given by

$$M_{ab} = \beta\,\mathrm{Re}\,m_{ab} + i\gamma_5\beta\,\mathrm{Im}\,m_{ab} = \begin{pmatrix} \mathrm{Re}\,m_{ab} & \mathrm{Im}\,m_{ab} \\ \mathrm{Im}\,m_{ab} & -\mathrm{Re}\,m_{ab} \end{pmatrix}, \tag{9.3}$$

where β is the Dirac metric and we have used the representation in which the rotation group is diagonal. There are no further conditions on λ, m and g unless we assume parity conservation. We do not assume parity conservation for the moment, partly for the sake of generality and simplicity, and partly because, even in the parity conserving case, we have to take into account the possibility that a spontaneous breakdown may occur in a parity non-conserving direction, that is, that there may be a spontaneous breakdown of parity.

Evaluating $\mathcal{L}$ in the usual way, we find

$$\mathcal{L} = \mathcal{L}_{k.e.} + \mathcal{F}_a^* \mathcal{F}_a + \Big\{ \lambda_a \mathcal{F}_a + m_{ab} \left(\mathcal{F}_a \mathcal{A}_b - \bar{\psi}_a \psi_b \right) + g_{abc} \left(\mathcal{F}_a \mathcal{A}_b \mathcal{A}_c - \bar{\psi}_a (A_b - \gamma_5 B_b) \psi_c \right) + h.c. \Big\}$$

where

$$\mathcal{F}_a = F_a + i G_a, \qquad \mathcal{A}_a = A_a + i B_a, \tag{9.4}$$

and hence we find that the potential V is given by

$$2V = \mathcal{F}_a^\dagger \mathcal{F}_a \qquad \text{where} \qquad -\mathcal{F}_a^* = \lambda_a + m_{ab} \mathcal{A}_b + g_{abc} \mathcal{A}_b \mathcal{A}_c . \tag{9.5}$$

We shall consider the possibility that spontaneous symmetry breaking can occur for any value $\mathcal{A}_a = z_a \neq 0$ of A_a and B_a. From (9.5) we see that any subsequent shift $\mathcal{A}_a \to \mathcal{A}_a' = \mathcal{A}_a - z_a$ to make the vacuum expectation value of the fields zero, can be absorbed in the parameters according to the identity

$$\mathcal{L}(\mathcal{A} + z, m, \lambda) = \mathcal{L}(\mathcal{A}, m(z), \lambda(z)), \tag{9.6a}$$

where

$$m(z) \equiv m_{ab}(z) = m_{ab} + 2 g_{abc} z_c , \tag{9.6b}$$

$$\lambda(z) \equiv \lambda_a(z) = \lambda_a + m_{ab} z_b + g_{abc} z_b z_c . \tag{9.6c}$$

We shall call the sets of tensors $m_{ab}(z), \lambda_a(z)$ for all complex numbers z the <u>orbits</u> of m_{ab} and λ_a.

The fact that the shift of the fields can be absorbed in the parameters, means that in the neighbourhood of any point $A_a = z_a$ the potential (9.5) takes the form

$$2V = \lambda_a^*(z)\lambda_a(z) + 2\mathrm{Re}\,\lambda_a^*(z)\, m_{ab}(z)\,\varepsilon_b + m_{ab}^*(z)\, m_{ac}(z)\,\varepsilon_b^*\varepsilon_c + 2\mathrm{Re}\, g_{ab}(z)\,\varepsilon_a\varepsilon_b , \qquad (9.7a)$$

where

$$\varepsilon_a = A_a - z_a \qquad \text{and} \qquad g_{ab}(z) \equiv g_{abc}\,\lambda_c^*(z). \qquad (9.7b)$$

If the point $A_a = z_a$ is a minimum, the linear term vanishes and the quadratic term may be written in the matrix form

$$\begin{pmatrix} \varepsilon^* & \varepsilon \end{pmatrix} \begin{pmatrix} \frac{1}{2} m^\dagger(z) m(z) & g^\dagger(z) \\ g(z) & \frac{1}{2} m(z) m^\dagger(z) \end{pmatrix} \begin{pmatrix} \varepsilon \\ \varepsilon^* \end{pmatrix} \qquad (9.8)$$

The matrix in (9.8) is therefore the boson mass-matrix at the minimum. By comparing (9.8) with (9.3) one can fairly easily see that there will be a spontaneous breaking of supersymmetry (boson and fermion masses unequal) if, and only if,

$$g(z) \equiv g_{ab}(z) = g_{abc}\,\lambda_c^*(z) \neq 0. \qquad (9.9)$$

Thus the matrix $g(z)$ controls the spontaneous symmetry breaking. In particular, if $V = 0$ at any point $A = z$, then $\lambda(z) = 0$, and hence $g(z) = 0$, at that point, and there is no spontaneous breaking.

To obtain a spontaneous breakdown of supersymmetry, we must therefore seek interactions g_{abc} such that $g_{ab}(z)$ will not be zero at the minimum of V. The results in that direction will be based on the following lemma which shows that such interactions are quite exceptional. Let us denote by Ω and $\Omega^{\perp}$ the zero and non-zero subspaces of $m(z)$ respectively $(\psi \in \Omega \Leftrightarrow m\psi = 0)$ and by $m_{\perp}$ and $g_{\perp}$ the restrictions of $m(z)$ and $g(z)$ to the non-zero subspace $\Omega^{\perp}$. Then the lemma is the following:

Lemma (a) Necessary and sufficient conditions for $A = z$ to be a local minimum of the potential are

(i) $\lambda^*(z) \in \Omega$ (ii) $g(z)\Omega = 0$ (iii) $m_{\perp}^{\dagger}(z) m_{\perp}(z) \geq 2|g_{\perp}(z)|$.

(b) A necessary condition for $A = z$ to be an absolute minimum is that $\det(m_{\perp} - \omega g_{\perp})$ be independent of ω, i.e.

$$\det(m_{\perp} - \omega g_{\perp}) = \det m_{\perp} \neq 0,$$

where ω is any complex number.

The first part of the lemma follows directly from (9.7) and is simply the requirement that the effective meson mass-matrix at $A = z$ be positive (that there be no tachyons). The second part of the lemma is less direct, but the point is that if $\det(m_{\perp} - \omega g_{\perp})$ depends on ω, we can solve $\det(m_{\perp} - \omega g_{\perp}) = 0$ for some $\omega = \mathring{\omega}$, and then, letting $z_a' = z_a + \frac{1}{2}\mathring{\omega}\lambda_a^*(z)$, show that $V(z) = V(z')$, where the point z' cannot be a minimum because condition (a) is not satisfied.

Note that an immediate corollary to (i) and (ii) of part (a) of the lemma is the following:

Corollary: If $A = z$ is a local minimum then

$$m_{ab}(z)\lambda_b^*(z) = 0 \quad , \quad g_{abc}\lambda_b^*(z)\lambda_c^*(z) = 0. \tag{9.10}$$

Indeed from the definition (9.7b) of $g(z)$ we see that (9.10) is simply the statement that $m(z)\lambda^*(z) = g(z)\lambda^*(z) = 0$ which is just (i) and a special case of (ii).

We shall relegate the formal proof of the lemma to Appendix C and consider here only its applications. The main application is, of course, to note that the second part of the lemma places very strong restrictions on the possibilities for spontaneous symmetry breaking, because in general a determinant of the form $\det(m_\perp - \omega g_\perp)$ is a polynomial in ω unless $g_\perp = 0$, in which case there is no spontaneous breakdown. To have $\det(m_\perp - \omega g_\perp)$ independent of ω for non-zero $g_\perp$, we must have $m_\perp$ and $g_\perp$ inter-related and $g_\perp$ degenerate (in particular $\det g_\perp = 0$). We shall see later that such a situation can arise, but first we consider some general and natural situations in which the lemma completely forbids a breakdown of supersymmetry.

The first such situation is when the full mass-matrix $m_{ab}(\mathring{z})$ is positive definite for at least one point $\mathring{z}$ on its orbit. This is the situation for the case of one scalar superfield described in the last section, and for the case when the Lagrangian is invariant with respect to an irreducible representation of an internal symmetry group $\left(m_{ab}(\mathring{z}) = m(\mathring{z})\,\delta_{ab}\right)$. To establish this result we actually only need the corollary to part (a) of the lemma, as follows: From the definition of $m_{ab}(z)$ we have

$$m_{ab}(\mathring{z}) = m_{ab}(z) + 2g_{abc}(\mathring{z}-z)_c\,. \tag{9.11}$$

Hence, using the corollary we obtain

$$m_{ab}(\mathring{z})\lambda^*_a(z)\lambda^*_b(z) = m_{ab}(z)\lambda^*_a(z)\lambda^*_b(z) + 2g_{abc}\lambda^*_a(z)\lambda^*_b(z)(\mathring{z}-z)_c = 0. \tag{9.12}$$

Since $m_{ab}(\mathring{z})$ is assumed positive definite we then have

$$\lambda^*(z) = 0 \tag{9.13}$$

Hence, in particular, $g(z) = 0$, and there is no spontaneous breakdown of supersymmetry.

There may be, of course, a spontaneous breakdown of other symmetries, such as an internal symmetry, for positive definite $m_{ab}(\mathring{z})$, and in that case it is interesting to note that (9.13) gives a quadratic vector equation for the location of the minima, in terms of the input parameters, namely,

$$g_{abc}(z_b-\mathring{z}_b)(z_c-\mathring{z}_c)+m_{ab}(\mathring{z})(z_b-\mathring{z}_b)+\lambda_a(\mathring{z})=0, \tag{9.13a}$$

where, in particular, if we take $\mathring{z}=0$ we obtain

$$g_{abc}z_b z_c + m_{ab}z_b + \lambda_a = 0. \tag{9.13b}$$

The result that a spontaneous breakdown of supersymmetry can only happen when $m_{ab}(z)$ is not positive definite at <u>any</u> point on the orbit is, of course, not to be confused with the result that $m_{ab}(z)$ is not positive definite at the minimum point, which is a trivial consequence of the Goldstone theorem. What we have shown here is that, if we are to have a spontaneous breakdown, the <u>input</u> mass matrix cannot be positive definite.

The other situations which we shall consider involve parity conservation. As explained in Appendix D, the phases of the superfields can be chosen so that the parity operator is defined by

$$(P\Psi_{\pm})(x\theta) = \Psi_{\mp}(x^P, i\gamma_0\theta), \tag{9.14}$$

and the intrinsic parity of the superfields is

$$\eta_a = \operatorname{sgn} m_a, \qquad m_{ab} = m^*_{ab} = m_a\delta_{ab}. \tag{9.15}$$

From Appendix D , a sufficient condition for parity conservation is simply

$$\lambda^*_a = \lambda_a, \qquad g^*_{abc} = g_{abc}, \tag{9.16}$$

that is, the criterion for parity conservation is the reality of the parameter. Note that (9.16) is maintained only under real shifts $A \to A + x$ of the fields A, so that (9.16) cannot be assumed for all points on the complex orbit. We shall assume it only at particular points such as the origin $\zeta = 0$ or an absolute minimum of the potential.

We now show that if, at a given absolute minimum, the parity is conserved and all the massive fields have the same intrinsic parity ($m_{ab} \geq 0$) there is no spontaneous breakdown of supersymmetry. Indeed in this case the condition (b) of the lemma is that $\det(m_\perp - \omega g_\perp)$ is independent of ω , where $m_\perp$ and $g_\perp$ are real and therefore hermitian, and $m_\perp$ is positive. But in that case $\det(1 - \omega\nu)$ in independent of ω , where ν is the hermitian matrix $m_\perp^{-\frac{1}{2}} g_\perp m_\perp^{-\frac{1}{2}}$, and diagonalizing ν , we see that this is possible only for $\nu = 0$, in which case $g = g_\perp = 0$, and there is no spontaneous breakdown.

Finally we consider counter-examples. Still taking the parity conserving case λ, m, g real as input, we note that the lemma and the result obtained above suggest that in order to find a counter-example we should let

$$m_{ab} = \left(\begin{array}{c|c} m_{ij} & 0 \\ \hline 0 & 0 \end{array}\right), \quad \lambda_a = \left(\begin{array}{c} 0 \\ \hline \lambda \end{array}\right), \tag{9.17}$$

where m_{ij} is non-singular, but indefinite, and the simplest such possibility is clearly

$$m_{ab} = \left(\begin{array}{cc|c} m_1 & 0 & 0 \\ 0 & -m_2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \quad \lambda_a = \left(\begin{array}{c} 0 \\ 0 \\ \hline \lambda \end{array}\right). \tag{9.18}$$

Since it is the projection $g_{ab} = g_{abc}\lambda_c$ of g_{abc} that is important for stability, we then concentrate on this projection by letting

$$g_{abc} = \left(g_{ab}\lambda_c + g_{bc}\lambda_a + g_{ca}\lambda_b\right) / \lambda^2 \tag{9.19}$$

where

$$g_{ab} = g \ell_a \qquad \text{OR} \qquad g_{ab} = \left(\begin{array}{cc|c} g_1 & g_3 & 0 \\ g_3 & g_2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right). \tag{9.20}$$

$$g_{ab}\lambda_b = 0$$

Condition (a) of the lemma is then met if m_{ij}^2 dominates g_{ij}, which is obviously true for sufficiently large m_1, m_2 relative to g_1, g_2, g_3. To meet condition (b) we note from (9.18) and (9.20) that

$$\det(m_\perp - \omega g_\perp) = \omega^2 \det g_\perp + \omega(m_1 g_2 - m_2 g_1) - m_1 m_2, \tag{9.21}$$

and that this expression is independent of ω, if and only if,

$$\frac{m_1}{m_2} = \frac{g_1}{g_2}, \qquad g_3^2 = g_1 g_2. \tag{9.22}$$

Thus the condition can be met, but only if g_{ab} is essentially uniquely determined by m_{ab}, that is

$$g_{ab} = \frac{g}{m_1+m_2}\left(\begin{array}{cc|c} m_1 & \sqrt{m_1 m_2} & 0 \\ \sqrt{m_1 m_2} & m_2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right), \tag{9.23}$$

which is fixed up to an overall constant g and the sign of the square-root. We normalize the overall constant so that $\mathrm{tr}\, g_\perp = g$.

Thus with λ_a, m_{ab}, g_{abc} given by (9.18), (9.19) and (9.23) the lemma does not exclude spontaneous symmetry breaking. This does not mean, however, that a spontaneous breakdown actually occurs, because part (b) of the lemma gives only a necessary condition for its occurrence. However, one can show that in the present case a spontaneous breakdown actually does occur. To show this one simply uses the expression (9.5) for the potential, with m_{ab} and g_{abc} as in (9.13), (9.19) and (9.23) to obtain

$$2V = \lambda^2 + (m^2+2\lambda g)A_1^2 + (m^2-2\lambda g)B_1^2 + m^2(A_2^2+B_2^2) + g^2(A_1^2+B_1^2)^2 + 4g^2(A_1^2+B_1^2)(A_0^2+B_0^2) + 2mg\left[A_0(A_1A_2+B_1B_2)+B_0(A_1B_2-B_1A_2)\right].$$

This potential satisfies $V \geq \lambda^2/2$ for $m^2 \geq \pm 2\lambda g$, and attains the minimum $\lambda^2/2$ at the origin, where there is clearly a spontaneous breakdown of supersymmetry since $\lambda g \neq 0$. One can also verify that with suitable parity assignments, parity is conserved both before ($\lambda = 0$) and after ($\lambda \neq 0$) symmetry breaking.

Another example has been constructed independently by P. Fayet, using as interaction Lagrangian density the highest weight of the expression

$$\hat{\mathcal{L}}_{int} = \lambda\Phi_+ + g\bar{\Psi}_+\Psi_+\Phi_+ + f\bar{\Psi}_+\vec{\tau}\Psi_+\vec{\Phi}_+ + h.c. \tag{9.25}$$

where Φ, $\vec{\Phi}$ and Ψ are scalars, vectors and spinors with respect to internal SU(2) ⊗ U(1) symmetry, and the expression (9.25) is the most general renormalizable one which is invariant with respect to this symmetry and the R-symmetry

$$\Phi(x,\theta) \to e^{2i\alpha}\Phi(x, e^{-i\gamma_5\alpha}\theta), \quad \vec{\Phi}(x\theta) \to e^{2i\alpha}\vec{\Phi}(x, e^{-i\gamma_5\alpha}\theta), \quad \Psi(x\theta) \to \Psi(x, e^{-i\gamma_5\alpha}\theta), \tag{9.26}$$

which is similar to the gauge symmetry C(1) of section 4. One can easily check that in this model also V is strictly positive, that there is a spontaneous breakdown of supersymmetry (though not at the origin), and that parity is conserved before and after the breakdown.

Since the SU(2) ⊗ U(1) and R-invariance of this model guarantee that the form of the interaction is not changed after renormalization, it might be worthwhile to mention that the first model also turns out to be R-invariant. In fact, it is the most general renormalizable interaction which is invariant with respect to the R-group (with Φ_1 the R-scalar) and the trivial internal symmetry group $\Phi_0 \to \Phi_0$, $\Phi_{1,2} \to -\Phi_{1,2}$.

Finally it should be mentioned that both the above models suffer from the peculiarity that at the potential minimum the values of all the fields are not fixed. For example, in the first model, one sees from (9.24) that V takes the minimal value $\frac{1}{2}\lambda^2$ at $A_{1,2} = B_{1,2} = 0$ for any value of A_0 and B_0. Thus the result that the supersymmetry is spontaneously broken, and also that the parity is conserved, holds in the first instance only at the classical level (tree approximation). The <u>effective</u> QFT potential, which includes the radiative corrections, might alter the situation quite drastically.

10. Other Possibilities for Mass-Breaking.

Since the spontaneous symmetry breaking for chiral scalar superfields demands the use of an input mass-matrix with at least one supermultiplet of mass zero, it might be worthwhile to consider other possibilities for splitting the boson and fermion masses without losing the 'good' properties of supersymmetry. Two such possibilities are:

(1) Modification of the basic supersymmetric algebra so that $P_\mu P^\mu$ would no longer be an invariant.

(2) Ad hoc insertion of mass-terms such as $\frac{1}{2}\Delta\mu^2 A^2$, $\frac{1}{2}\Delta\sigma^2 B^2$ or $\frac{1}{2}\Delta m\bar{\psi}\psi$.

The first possibility would clearly be the most attractive of the two. However, it turns out not to be feasible. In fact, the proposal runs into exactly the same difficulties as did similar proposals in the case of Lie algebras.(16) What happens is that no matter how the enlarged algebra is constructed, so long as it is finite, the mass-operator is nilpotent with respect to commutation, and so long as the mass-operator is nilpotent, in any unitary irreducible representation the mass-spectrum must be continuous or consist of a single point. As this seems to be a no-go theorem which survives the transition from commutators to anti-commutators, it might be worthwhile to sketch the old proof in the new context: We start from the assumption that we have a <u>finite</u> set of operators T_a, $a=1\ldots n$, that is, a <u>finite</u> supersymmetric algebra, including the Poincaré generators $L_{\mu\nu}$, P_μ in conventional notation, and that whatever the commutation or anti-commutation relations of the T_a among themselves (or even the lack of such relations), we at least have

$$[L_{\mu\nu}, T_a] = i C^{ab}_{\mu\nu} T_b, \qquad [P_\mu, T_a] = i C^{ab}_\mu T_b. \tag{10.1}$$

In other words, we assume that the algebra T_a forms a finite-dimensional tensor with respect to the Poincaré group. Then the consistency of (8.5) with usual commutation relations for the Poincaré algebra will mean that the $C^{ab}_{\mu\nu}$, C^{ab}_{μ}, considered as matrices $C_{\mu\nu}$, C_μ with indices a, b, form a finite-dimensional representation of the Poincaré group. In that case the commutation properties of the $L_{\mu\nu}$, P_μ can be transferred to the $C_{\mu\nu}$, C_μ and vice versa. We can therefore make the transfer

$$i[L_{\mu\nu}P^{n-1}_\nu, P_\mu] = P^n_\nu \rightarrow i[C_{\mu\nu}C^{n-1}_\nu, C_\mu] = C^n_\nu, \quad \mu \neq \nu. \tag{10.2}$$

But since the C-algebra is finite-dimensional, and the trace of a finite dimensional matrix is zero, it follows from (10.2) that $\mathrm{Tr}\, C^n_\nu$ is zero. It follows that the C_μ are nilpotent matrices, and, since they commute, that $C_\mu C^\mu$ is a nilpotent matrix, i.e. there exists an integer N such that $(C_\mu C^\mu)^N$ is zero. Then transferring this property back to the P_μ we see that for the integer N, we have

$$\delta^N(P^2)T_a = 0, \quad \text{where } \delta(P^2)X \equiv [P^2, X], \quad P^2 = P_\mu P^\mu, \tag{10.3}$$

i.e. that the mass-operator is nilpotent with respect to commutation. If we now assume that we have particle states with discrete masses m_β and that the T_a are well-defined on these states, then sandwiching (10.3) between two such states, we get

$$(m^2_\alpha - m^2_\beta)^N \langle\alpha|T_a|\beta\rangle = 0. \tag{10.4}$$

This shows that the T_a cannot connect two states with different masses, so that in each irreducible subspace of this kind there can be only one mass. The argument is not rigorous because the T_a might not be well-defined on the states $|\alpha\rangle$. However, if at least one of the masses is isolated (i.e. if there are no zero-mass particles present) then under standard technical conditions (essentially that there exist an invariant dense domain on which the algebra is well-defined and at the same time $L_{\mu\nu}, P_\nu$ can be exponentiated to a unitary representation of the Poincaré group) equation (9.7) itself can be used to show that the $|\alpha\rangle$ do indeed lie in the right domain, so that the simple argument given is justified. (In any case, if the physical states $|\alpha\rangle$ did not lie in the domain of the supersymmetric operators T_a the algebra would lose its direct physical meaning.)

The conclusion is, therefore, that a supermultiplet of particles of different non-zero discrete masses cannot be constructed with a finite-dimensional algebra of any kind which contains the Poincaré algebra as a subalgebra, and this includes any supersymmetric algebra as a special case. It should be emphasized however that this result does not apply to spontaneous symmetry breaking, for which there are zero-mass (Goldstone) particles, and the T_a are not operators in the usual sense.

We now consider the second possibility, namely the insertion of ad hoc mass terms of the form $\frac{1}{2}\Delta\mu^2 A^2$, $\frac{1}{2}\Delta\sigma^2 B^2$ or $\frac{1}{2}\Delta m\bar{\psi}\psi$. For this purpose we write down a more general Lagrangian of the WZ-type, i.e. a WZ-type Lagrangian <u>without</u> the WZ mass and coupling constant correlations

$$\mathcal{L} = \mathcal{L}_{k.e.} + \underset{(-2)}{\tfrac{1}{2}\mu^2 A^2} + \underset{(-2)}{\tfrac{1}{2}\sigma^2 B^2} + \underset{(-3)}{\tfrac{1}{2}m\bar{\psi}\psi} + \underset{(-4)}{g\bar{\psi}(A-\gamma_5 B)\psi} + \underset{(-3)}{fA(A^2+B^2)} + \underset{(-4)}{\tfrac{1}{2}G^2(A^2+B^2)^2} \qquad (10.5)$$

where, underneath each term in the Lagrangian we have also written its dimension. The completely supersymmetric case is then characterized by the inequalities

$$\mu = \sigma = m \qquad f = mg \qquad G = g \qquad (10.6)$$

and the crucial point to recall is that these equalities persist after renormalization. The dimensional argument that the insertion of a term of a given dimension affects the renormalization of terms only of the same or higher dimension, would then indicate that the softest breaking would be through the insertion of a term of the form $\frac{1}{2}\Delta\mu^2 A^2 + \frac{1}{2}\Delta\sigma^2 B^2$. According to the dimensional argument such an insertion would break the first equality in (9.10), but would not affect the other two equalities, even after renormalization. The insertion of a term of the form $\frac{1}{2}\Delta m \bar{\Psi}\Psi$ would be less soft, and would preserve only the last equality in (9.10) after renormalization. The preservation of this equality would, however, still be enough to eliminate the linear and quadratic divergences. In any case the softest mass-breaking Ansatz would appear to be

$$\mathcal{L} = \mathcal{L}_{k.e.} + \tfrac{1}{2}\mu^2 A^2 + \tfrac{1}{2}\sigma^2 B^2 + \tfrac{1}{2} m \bar{\Psi}\Psi + g\bar{\Psi}(A - \gamma_5 B)\Psi + mgA(A^2+B^2) + \frac{g^2}{2}(A^2+B^2)^2 . \tag{10.7}$$

This Ansatz is also supported by the results of section 9 where, after spontaneous symmetry breaking, it is the fermion mass which appears in the coupling constant for the cubic boson interaction.

The question now is whether the Ansatz (10.7) is maintained after renormalization. This is certainly true in the second order approximation, since in this approximation the vertex, charge and wave-function renormalizations are the same as in the symmetric limit (up to finite corrections). Furthermore, in the 2nd order approximation, and up to finite corrections, we obtain for the masses (Appendix E),

$$\overset{(2)}{\Delta}(\mu^2+\sigma^2-2m^2) = 4g^2(\mu^2+\sigma^2-2m^2)I \quad , \qquad \overset{(2)}{\Delta}(\sigma^2-\mu^2) = 4g^2(\sigma^2-\mu^2)I \tag{10.8}$$

Thus either of the mass formulae

$$\mu^2+\sigma^2 = 2m^2 \qquad\qquad \mu^2 = \sigma^2 \tag{10.9}$$

or equivalently, the general mass-formula

$$\alpha\mu^2 + \beta\sigma^2 + \gamma m^2 = 0 , \qquad \alpha+\beta+\gamma = 0 . \tag{10.10}$$

can also be maintained in this approximation. The case of ad hoc mass-breaking with the first mass formula in (10.9) has been considered in great detail by Iliopoulos and Zumino[12], who have shown that this formula can be maintained (up to finite corrections) to <u>all</u> orders in the coupling constant.

11. Internal Symmetry and Supersymmetry.

It is natural to wish to introduce internal symmetry into supersymmetry. If one wishes to introduce it in the direct product form then there is very little difficulty.[10][17] For example, if we let the superfields $\Psi^a_\pm$ transform according to some global internal symmetry group such as SU(2), $\Psi^a_\pm \to S_{ab}\Psi^b_\pm$ then the free part of the Lagrangian (5.1) generalizes at once to

$$\mathcal{L} = -\frac{1}{8} h\{\Psi^\dagger_+\Psi_+ + \Psi^\dagger_-\Psi_-\} + \frac{m}{2} h\{\Psi^\dagger_+\Psi_- + \Psi^\dagger_-\Psi_+\}, \qquad S^\dagger S = 1. \tag{11.1}$$

where h denotes the highest weight and dagger denotes transpose in the representation space as well as complex conjugation. Interactions are a little more restrictive because renormalizability restricts them to being trilinear, and it is not always possible to construct trilinear group invariants with a single representation. However, if the representation S is real and has non-trivial trilinear coupling, then the obvious generalization of the WZ interaction in (5.1) is

$$\mathcal{L}_g = g\, d_{abc}\{\Psi^a_+\Psi^b_+\Psi^c_+ + \Psi^a_-\Psi^b_-\Psi^c_-\}, \qquad \Psi^{*a}_\pm = \Psi^a_\mp, \tag{11.2}$$

where d_{abc} is the real trilinear coupling. The only simple compact Lie groups for which this can happen are SU(n) for n ⩾ 3, but it can also happen for semi-simple groups of the form $G \otimes G$ where G is a compact simple Lie group (since the product of two totally anti-symmetric F-couplings is a totally symmetric D-coupling) and it may happen also for finite-dimensional groups such as the cyclic group $\Psi^1 \to \Psi^2 \to \Psi^3 \to \Psi^1$. On the other hand, if the representation S is not real we can always introduce other fields $\Phi_\pm$ belonging to some representation Δ Δ of the group which occurs in the decomposition $S^* \otimes S$, adjoin to (11.1) a similar kinetic term for $\Phi_\pm$, and then construct an interaction of the form

$$\mathcal{L}_g = g\{\Psi^{*a}_- \Phi^{ab}_- \Psi^b_+ + \Psi^{*a}_+ \Phi^{ab}_+ \Psi^b_-\}. \tag{11.3}$$

For example, if S were the fundamental representation of SU(N), Φ could be the scalar or the adjoint representation. The analogy between (11.3) and the π- N and σ- N interaction is obvious.

One might now ask, however, whether a <u>non-trivial</u> mixing of internal symmetry and supersymmetry is possible, since the usual no-go theorems which inhibit such mixing depend heavily on the fact that the internal group is a compact <u>Lie</u> group, and so might not apply. It turns out that non-trivial mixing is indeed possible,[18] and can be introduced by assigning an internal symmetry label, not to the superfield itself as above, but to the parameters θ within the superfield. That is, one lets the superfield be

$$\Phi(x,\theta^a)\,, \qquad \{\theta^a_\alpha, \theta^b_\beta\} = 0 \qquad \theta^* = \eta\theta \tag{11.4}$$

where a is an internal symmetry index, and the internal representation S is assumed to be self-conjugate in order to satisfy the Majorana condition. That is, $S^* = \eta S \eta^{-1}$ where η is a unitary intertwining matrix. Since S is unitary it must then be either orthogonal or symplectic, the latter case including the two-dimensional representation of SU(2). The quantities $\frac{i}{2}\bar{\theta}\Gamma\gamma_\mu\epsilon$ where $\Gamma = 1$ and $\Gamma = \gamma_5$ for the orthogonal and symplectic cases respectively, are then real, and so the natural generalization of the supersymmetric transformation law is simply

$$\left(U(\epsilon)\Phi\right)(x_\mu,\theta) = \Phi\left(x_\mu + \tfrac{i}{2}\bar{\theta}\Gamma\gamma_\mu\epsilon,\ \theta+\epsilon\right) . \tag{11.5}$$

If one now expands the superfield $\Phi(x,\theta)$ in powers of θ^a_α, one sees at once that one obtains a non-trivial mixing of internal and spin symmetry. For example, for real θ $(\eta = 1)$ the bilinear term of (1 .2) becomes

$$F^{\underline{ab}} + \gamma_5 G^{\underline{ab}} + i\gamma_5 \not{B}^{\underline{ab}} + \not{C}^{ab} + i\sigma_{\mu\nu} C^{ab}_{\mu\nu} \ . \tag{11.6}$$

where $\underline{ab}$ and $\underset{\vee}{ab}$ are symmetric and anti-symmetric respectively. Thus if θ belongs to the three-dimensional representation of the internal symmetry group SU(2), the Lorentz scalar, pseudo-scalar and pseudo-vector belong to the T = 0, 2 representations, and the Lorentz vector and anti-symmetric tensor to the three-dimensional representation.

This result is extremely interesting because the supersymmetric transformation law (11.5) is completely compatible with Lorentz covariance, and hence the Ansatz (11.4) represents the first fully relativistic self-consistent mixing of internal and Lorentz symmetry. In practice the Ansatz, as presently made, runs into the difficulty that its representations are too large. In general, if n is the dimension of the internal representation S, then because the 'Lie' algebra

$$\{G^a_\alpha, G^b_\beta\} = \eta^{ab} \, (\Gamma C \gamma)_{\alpha\beta} , \tag{11.7}$$

of (11.5) reduces to a Clifford algebra in the rest-frame, it is 2^{4n} - dimensional. Thus the smallest non-trivial dimension is 256. By making a parity projection similar to the chiral projection of section 4 one can reduce the number of Dirac indices effectively to two and hence improve 2^{4n} to 2^{2n}. The smallest dimension is then 16, and for this case one has obtained the (spin-isospin) parity correlations [18]

$$(0,0)^+ + (\tfrac{1}{2},\tfrac{1}{2})^i + (1,0)^- + (0,1)^- + (\tfrac{1}{2},\tfrac{1}{2})^{-i} + (0,0)^+ . \tag{11.8}$$

Unfortunately this multiplet does not correspond to any known set of particles of equal, or approximately equal, mass. Hence there still remains the problem of making the Ansatz (11.4) realistic or of finding a more appropriate Ansatz.

A general investigation of this problem has been undertaken, and I am indebted to Dr. Sohnius for information concerning some early results [19] which are already quite restrictive. What one finds is that if one tries to combine supersymmetry non-trivially with a compact internal symmetry, one has the following restrictions:

(i) All spinor charges commute with the momentum, $[P_\mu, G] = 0$.

(ii) All spinor charges are spin $\frac{1}{2}$, $G = G_a$, $a = 1, 2$

(iii) The most general anti-commutation relations are

$$\{G^m_a, \bar{G}^{\dot{m}}_{\dot{a}}\} = \delta_{m\dot{m}} \sigma^\mu_{a\dot{a}} P_\mu, \tag{11.9}$$

$$\{G^m_a, G_b\} = \varepsilon_{ab} d^s_{\underset{\vee}{mn}} C_s, \quad [d^s_{mn} C_s, G] = 0,$$

where the C_s are the internal symmetry generators and the $d^s_{\underset{\vee}{mn}}$ are C-G coefficients. In the case that the internal symmetry is self-conjugate, we can define the Majorana spinor θ_α, and in terms of this spinor (11.9) reduces to

$$\{G^m_\alpha, G^n_\beta\} = \eta^{mn} (\Gamma C p)_{\alpha\beta} + (\Gamma C)_{\underset{\vee}{\alpha\beta}} A_{\underset{\vee}{mn}} + (\Gamma C \gamma_5)_{\underset{\vee}{\alpha\beta}} B_{\underset{\vee}{mn}} \tag{11.10}$$

where

$$A + iB = C \equiv d^s_{\underset{\vee}{mn}} C_s$$

and

$$[C, A] = [G, B] = 0.$$

Eq. (11.10) is clearly only a slight generalization of (11.7).

12. Spontaneous Breakdown of Internal Symmetry.

In section 9 we saw that for chiral scalar superfields the supersymmetry cannot be spontaneously broken if the input mass-matrix is positive definite. In this section we shall show that although in the positive definite case the supersymmetry itself is not spontaneously broken, nevertheless it has the remarkable property that it acts as a catalyst for the spontaneous breakdown of internal symmetry.[10] In fact from the results of section 9 we can show at once that the addition of supersymmetry to ordinary symmetry in direct product form produces a Goldstone potential for the spontaneous breakdown of the internal symmetry. (This may happen even if the mass-matrix is not positive definite, but we shall confine ourselves here to the positive definite case.)

First, we recall from section 9 that if the input mass-matrix is positive definite the potential minima are at V = 0 , and hence lie at the points where the z_a are given by the quadratic vector equation (9.13b), and that at these minima the effective mass-matrix is given by (9.6b). This result holds for any set of chiral scalar superfields. Let us now consider the special case where the chiral scalar superfields belong to a real irreducible representation R of some group with respect to which the Lagrangian is invariant. Then the parameters λ_a, m_{ab} and g_{abc} are invariant tensors with respect to that group. For simplicity (and for relevance) let us now suppose that G is a semi-simple compact group, or, more generally, the product of such a group with a discrete group such as the permutation or reflexion group, and let us assume that the completely symmetric part of RxRxR contains the trivial representation (with multiplicity one). We then have

$$\lambda_a = 0 , \qquad m_{ab} = m\delta_{ab} , \qquad g_{abc} = g\, d_{abc} . \tag{12.1}$$

where the $d_{abc} \neq 0$ are the completely symmetric C-G coefficients. From (9.13b) we then see that the parity-preserving potential minima are at the points $A_a = -\frac{m}{g} f_a$ where the f_a are the real solutions of the equation

$$d_{abc} f_b f_c = f_a , \tag{12.2}$$

and only at these points. At these minima we have the one-parameter mass-formula

$$m_{ab}(f) = m(\delta_{ab} - 2 d_{abc} f_c) . \tag{12.3}$$

for the effective mass. Note that the equation (12.2) is just the fixed point equation for the mapping $R \otimes R \to R$. Such equations have been considered as a symmetry breaking mechanism by a number of authors from different points of view[20] and an extensive mathematical investigation of the solutions has been carried out by Michel and Radicati.[21] In general the solutions f_a can lie only in discrete directions, such as the directions orthogonal to I, V and U spin in SU(3). This is in strong contrast to the non-supersymmetric case, where the internal symmetry alone would allow Goldstone potentials of the form $-\mu^2 A_a A_a + \lambda^2 (A_a A_a)^2$ and hence would allow breaking in any arbitrary direction.

If we recall that in the positive definite case the supersymmetry itself is not broken by the above mechanism, we see that what happens is that the internal multiplet splits up into submultiplets, each one of which is separately supersymmetric. The fermions and bosons in each submultiplet have the same mass, but the mass varies from one submultiplet to another.

Let us now consider the question of Goldstone particles[22] for the spontaneous breaking (12.3). For this purpose we recall the general Goldstone result that since the potential is group invariant ($V(A) = V(A + \delta g A)$ where δg is any generator) the quantity $\delta^2 V / (\delta g A)^2$ is zero, and hence, in particular at any potential minimum $A = \frac{m}{g} f$, the zero-mass, or Goldstone, directions are just the directions $\delta g f$. If we let δH denote the Lie algebra of the little group of f we have, by definition, $\delta H f = 0$ and $\delta g f \neq 0$, $\delta g \notin \delta H$, and so we see that the Goldstone directions are just the complement of the little group directions. (Another way to say this is that the Goldstone directions correspond in a one-to-one way with the cosets G/H.)

In our case therefore, the Goldstone directions are just $G_\alpha f_a$ where G_α are the group generators. This can also be verified directly from (12.2) and (12.3), since if we make an infinitesimal group transformation of (12.2) we obtain from the covariance of the equation and the invariance of the d_{abc}

$$2 d_{abc} f_b (Gf)_c = (Gf)_a , \tag{12.5}$$

and hence from (12.3) we see that

$$m_{ab} (Gf)_b = 0 . \tag{12.6}$$

as predicted.

It is interesting to note that the mass-formula (12.3) is independent of the normalization of f_a and d_{abc}. For, from (12.2) we see that a renormalization of f_a induces the inverse renormalization for d_{abc}, and conversely; and from (12.3) we see that any such renormalization leaves (12.3) invariant.

With a view to applications let us now consider some special cases. First, the only (23) compact simple groups whose adjoint representations have a non-zero totally symmetric D-coupling of the kind required are SU(N) for N ⩾ 3. For these the d_{abc} are defined by the equations

$$\{M_a, M_b\} - \frac{1}{N}\delta_{ab}\, \mathrm{tr}\{M_a, M_b\} = d_{abc} M_c \tag{12.7}$$

for the N x N trace-orthogonal matrices M_a of the fundamental representation. The fixed point equations (12.2) have been solved for this case by Michel and Radicati, using the fact that for the adjoint representation the vectors f_a may be replaced by the matrices of the fundamental representation so that (12.2) reduces to

$$M^2 - \frac{\mathrm{tr} M^2}{N} I = M, \quad \mathrm{tr} M = 0 , \quad M^\dagger = M . \tag{12.8}$$

For SU(3) the solutions of (12.8) are easily seen to be M = diag. (-1, -1, 2) (up to conjugation) and these are just the directions with little groups SU(2) $\otimes$ U(1) mentioned earlier. If we choose the conventional hypercharge direction, with isospin $\otimes$ hypercharge as little group, then we easily obtain from (12.3) the mass-formula

$$M(\tfrac{1}{2}) = 0 \qquad M(1)^2 = 9\, M(0)^2, \tag{12.9}$$

where the argument refers to isospin. In particular we note that the Goldstone particles are the I = $\frac{1}{2}$ members, in agreement with the general result that the Goldstone particles are just the complement of the little group particles, which here are I = 1 and I = 0.

Another particular case of interest is the set of adjoint $\otimes$ adjoint representations of G $\otimes$ G, where the G are any simple compact Lie groups. In this case the symmetric D-coupling is given by

$$d_{abc} = d_{i\alpha;\, j\beta;\, k\gamma} = f_{ijk} f_{\alpha\beta\gamma} \tag{12.10}$$

where the f_{ijk} are the completely anti-symmetric structure constants for G. Since

$$f_{ijk} f_{\alpha\beta\gamma} \delta_{j\beta} \delta_{k\gamma} = f_{i\beta\gamma} f_{\alpha\beta\gamma} = g_{i\alpha} = \kappa\, \delta_{i\alpha}, \tag{12.11}$$

where $g_{i\alpha}$ is the Killing form, it is clear that in this case one solution of the fixed point equation (12.2) is given by

$$f_{i\alpha} = \kappa\, \delta_{i\alpha}. \tag{12.12}$$

The little group for this solution is clearly the diagonal group G and the mass-formula is

$$m_{ab} = m_{i\alpha;\, j\beta} = m(\delta_{ij} \delta_{\alpha\beta} - 2 f_{ij\gamma} f_{\alpha\beta\gamma}). \tag{12.13}$$

According to the general results discussed above, the Goldstone fields in this case should correspond to the cosets G x G / Diagonal G ~ G. Hence there should be one Goldstone-field for each generator of G, and indeed it is easy to verify from the Jacobi identity that the generators f^{α} of G, where $(f^{\alpha})_{\beta\gamma} = f_{\alpha\beta\gamma}$, are eigenvectors of the mass-operator (12.13) with eigen-value zero.

For SU(2) $\otimes$ SU(2) the mass-formula (12.13) gives (10)

$$M(1) = 0, \qquad M(2)^2 = 4\, M(0)^2. \tag{12.14}$$

where the argument is the spin of the diagonal SU(2).

Finally, it may be well to emphasize that, since $V \geqslant 0$, if the points $V = 0$ are attained they are absolute minima. This circumstance allows us to look for general (not necessarily parity-conserving) minima by setting $\xi_a = 0$. From (9.4) we then have

$$\lambda_a + m_{ab} A_b + g_{abc} A_b A_c = 0, \tag{12.15}$$

or, equivalently,

$$\lambda_a + m_{ab} A_b + g_{abc}(A_b A_c - B_b B_c) = 0,$$
$$m_{ab} B_b + 2 g_{abc} A_b B_c = 0. \tag{12.16}$$

In particular, in the irreducible case we have

$$d_{abc}(A_b A_c - B_b B_c) = -\frac{m}{g} A_a$$
$$2 d_{abc} A_b B_c = -\frac{m}{g} B_b \tag{12.17}$$

To solve these equations one may use the procedure of Michel and Radicati mentioned previously, namely, one may define the hermitian matrices

$$A = -\frac{m}{g}\sum A_a \tau_a \qquad\qquad B = -\frac{m}{g}\sum B_a \tau_a \tag{12.18}$$

where τ_a are the group generators. Then, for SU(N) for example, (12.17) reduces to

$$A \vee A - B \vee B = A$$
$$2 A \vee B = B \tag{12.19}$$

where $2A\vee B$ is the anti-commutator of A and B minus its trace.

Equations (12.19) can be solved by noting that A commutes with B^2, and then diagonalizing these two matrices. For SU(3) the most general solution turns out to be

$$A = -\tfrac{1}{2}(q + 3 r_1 \cosh \alpha)$$
$$B = \tfrac{3}{2} r_2 \sinh \alpha \tag{12.20}$$

where α is an arbitrary real parameter, and the q and r_i, $i = 1, 2$ satisfy the algebra

$$q \vee q = q \qquad\qquad q \vee r_i = -r_i \qquad\qquad r_i \vee r_j = -\tfrac{1}{3}\delta_{ij}\, q \tag{12.21}$$

Here the q are just the real solutions (B=0) of (12.19) found earlier in this section, and hence they are the vectors orthogonal to U, V, and I-spin (charge-vectors), while the r_i are the real generators of the orthogonal SU(2) group, U, V, and I respectively. That is

$$q = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad r_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12.22}$$

up to conjugation.

13. Yang-Mills Superfields.

Before introducing Yang-Mills superfields, we recall the procedure for ordinary Yang-Mills fields:[24] Starting with a free fermion matter-field with Lagrangian

$$\mathcal{L}(\psi') = \mathcal{L}(\psi) = \bar{\psi}(i\not{\partial} - m)\psi , \qquad \psi'(x) = S\psi(x), \tag{13.1}$$

which is invariant with respect to a <u>global</u> internal symmetry group S such as SU(2), one modifies it so as to make it invariant with respect to the corresponding <u>local</u> group

$$\psi'(x) = S(x)\psi(x) , \qquad S(x) = e^{ie\Lambda(x)} , \qquad \Lambda(x) = \tau_\alpha \Lambda_\alpha(x) , \tag{13.2}$$

where τ_α are the group generators, by writing

$$\mathcal{L} = -\frac{1}{4}\, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{13.3}$$

where

$$D_\mu = \partial_\mu + ie\, B_\mu(x) \quad \text{and} \quad F_{\mu\nu} = -\frac{i}{e}[D_\mu, D_\nu] = \partial_\mu B_\nu - \partial_\nu B_\mu + ie[B_\mu, B_\nu] . \tag{13.4}$$

In other words, one introduces a new field B_μ whose transformation properties are chosen so as to make the derivative covariant i.e. so that

$$D'_\mu = S D_\mu S^{-1} . \tag{13.5}$$

The first term in (13.3) is then a kinetic term for the new field. The steps (13.1) to (13.3) are summarized in the first column of table 2 .

In the supersymmetric case the fields chosen as matter-fields are the chiral[10][25] superfields $\Psi_\pm$ (that is, ψ comes accompanied by the two scalar fields A and F) and the Lagrangian corresponding to (13.1) is the globally invariant Lagrangian

$$\mathcal{L} = -\frac{1}{8} h \left\{ \Psi_+^\dagger \Psi_+ + \Psi_-^\dagger \Psi_- \right\} + \frac{m}{2} h \left\{ \Psi_+^\dagger \Psi_- + \Psi_-^\dagger \Psi_+ \right\}, \tag{13.6}$$

of section 9. When we now consider the analogue of the local transformations (13.2), however, we see that if we are to preserve the supersymmetry and chirality, the gauge fields $\Lambda(x)$ must depend on θ as well as x , and must themselves have definite chirality. Thus in the supersymmetric case we must have

$$\Psi'_\pm(x\theta) = S_\pm(x\theta)\, \Psi_\pm(x\theta) , \qquad S_\pm(x\theta) = e^{ie\Lambda_\pm(x\theta)} , \qquad \Lambda_\pm(x\theta) = \Lambda^\alpha_\pm(x\theta)\, \tau_\alpha . \tag{13.7}$$

Thus in the supersymmetric case the single gauge field $\Lambda(x)$ is replaced by two independent gauge fields $\Lambda_\pm(x\theta)$. Furthermore since the $\Lambda_\pm(x,\theta)$ cannot be real, neither gauge transformation is unitary in itself:

$$\Lambda_\pm(x\theta) \neq \Lambda^*_\pm(x\theta) \qquad \Longrightarrow \qquad S^\dagger_\pm(x\theta)\, S_\pm(x\theta) \neq 1 . \tag{13.8}$$

The most that we can require is that

$$\Lambda_\pm(x\theta) = \Lambda^*_\mp(x\theta) \qquad \Longrightarrow \qquad S^\dagger_\pm(x\theta)\, S_\mp(x\theta) = 1, \tag{13.9}$$

and this condition we shall, in fact, assume throughout.

If we now apply the local transformations $S_\pm(x\theta)$ to the Lagrangian (13.6) and use (13.9) we obtain

$$\mathcal{L}' = -\frac{1}{8} h \left\{ \Psi_+^\dagger e^{-i\Lambda_-} e^{i\Lambda_+} \Psi_+ + \Psi_-^\dagger e^{-i\Lambda_+} e^{i\Lambda_-} \Psi_- \right\} + \frac{m}{2} h \left\{ \Psi_+^\dagger \Psi_- + \Psi_-^\dagger \Psi_+ \right\}. \tag{13.10}$$

Thus, as in the case of the conventional matter-field ψ , the mass-term in the Lagrangian remains invariant under the local transformation but the kinetic term does

(10)(25)

not. The invariance is restored as in the ordinary case by the introduction of a new field with suitable transformation properties. This time, however, the new field is a real (non-chiral) scalar supermultiplet $\Phi(A, \psi, F, G, B_\mu, \chi, D)$ in which the vector field B_μ plays the role of the conventional vector field. This field is inserted in the Lagrangian in the simplest way that will restore the invariance, namely, by writing

$$\mathcal{L} = -\frac{1}{8} h\left\{ \Psi_+^{\dagger} e^{ie\Phi} \Psi_+ + \Psi_-^{\dagger} e^{-ie\Phi} \Psi_- \right\} + \frac{m}{2} h\left\{ \Psi_+^{\dagger} \Psi_- + \Psi_-^{\dagger} \Psi_+ \right\}, \tag{13.11}$$

where the new field is assumed to have the transformation law

$$e^{ie\Phi'} = e^{ie\Lambda_-} e^{ie\Phi} e^{-ie\Lambda_+} \tag{13.12}$$

(Note that (13.12) reduces to $\Phi' = \Phi + \Lambda_- - \Lambda_+$ in the abelian case.) The only problem is that one has to construct a kinetic term for the new field, and this term must be supersymmetric, gauge-invariant (i.e. invariant with respect to (13.12)) and contain a term $\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$ in analogy to (13.3). The gauge invariance suggests that the term be formed from $e^{ie\Phi}$, and the supersymmetry that it be formed with the covariant derivatives $\mathcal{D}_\alpha$ of section four. The term $\mathrm{tr}\, F_{\mu\nu} F^{\mu\nu}$ as highest weight suggests a product $\ldots + \frac{1}{32}(\bar\theta \cdot \theta)^4 F_{\mu\nu} F^{\mu\nu}$, which suggests factors of the form $\frac{1}{4}\bar\theta \cdot \gamma_5 \gamma_\mu \theta F_{\mu\nu}$, and (recalling that scalar chiral fields have the terms $\frac{i}{4}\bar\theta \cdot \gamma_5 \gamma_\mu \theta B_\mu$) these factors in turn suggest <u>vector</u> chiral superfields Φ_μ. With all these conditions
(10)(25)
in mind one can then make the <u>Ansatz</u>

$$\mathcal{L}_{k.e.} = -\frac{1}{8} h\left\{ \Phi_\mu^{+} \Phi_\mu^{+} + \Phi_\mu^{-} \Phi_\mu^{-} \right\} \tag{13.13}$$

where

$$\Phi_\mu^{\pm} = \frac{1}{e}\left(C\gamma_\mu \frac{1 \pm \gamma_5}{2} \right)^{\alpha\beta} \mathcal{D}_\alpha \left(e^{-ie\Phi} \mathcal{D}_\beta e^{ie\Phi} \right), \tag{13.14}$$

and this Ansatz does indeed lead to a suitable kinetic term. Thus the full supersymmetric Yang Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{8}h\left\{\Phi^+_\mu \Phi^+_\mu + \Phi^-_\mu \Phi^-_\mu\right\} - \frac{1}{8}h\left\{\Psi_+^\dagger e^{ie\Phi}\Psi_+ + \Psi_-^\dagger e^{-ie\Phi}\Psi_-\right\} + \frac{m}{2}h\left\{\Psi_+^\dagger\Psi_- + \Psi_-^\dagger\Psi_+\right\} \tag{13.15}$$

and the steps leading to its construction are summarised in the second column of table 2.

The Lagrangian (13.15) is, of course, non-polynomial. However, because of the nilpotency of the θ_α it is actually only non-polynomial in the lowest weight field A, and in fact there exists a special gauge in which the Lagrangian not only becomes polynomial but effectively cubic. The existence of such a gauge can be seen by first considering the abelian case for which the gauge transformation (13.12) becomes $\Phi \to \Phi + \Lambda_- + \Lambda_+$ and noting from this equation that the super-gauge fields can be chosen to eliminate some of the conventional fields in Φ. To see which fields can be eliminated we recall Fig. 5 which when subtracted from Fig. 1 gives

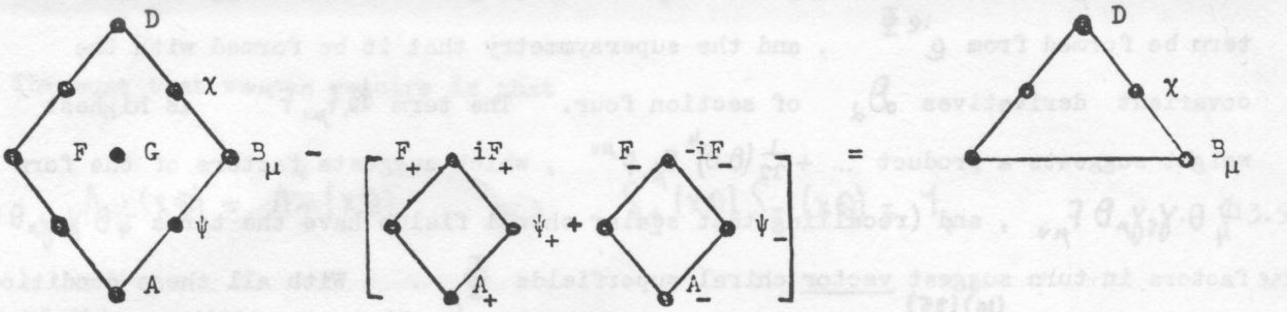


Fig. 13

From this figure we see at once that the fields A, ψ, F, G can be eliminated. Furthermore, we still have the conventional gauge freedom of eliminating the longitudinal part of B_μ with the B-field. Once the A, ψ, F, G fields are zero, the nilpotency of the θ's makes the expansion of the exponential terminate almost immediately to give

$$e^{ie\Phi} = 1 + ie\Phi - \frac{e^2}{16}(\theta\cdot\theta)^2 B_\mu B_\mu . \tag{13.16}$$

As one might expect, by modifying the gauge transformation to take account of the non-commutativity, the same result can be shown to hold in the non-abelian case also. Inserting the expansion (13.16) into the supersymmetric Yang-Mills Lagrangian (13.15) (10)(25) one obtains

$$\mathcal{L} = \text{const.}\ \mathrm{tr}_R\left\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\bar{\lambda}\not{D}\lambda + \tfrac{1}{2}d^2\right\} - \tfrac{1}{2}(D_\mu A)^\dagger(D_\mu A) - \tfrac{1}{2}(D_\mu B)^\dagger(D_\mu B) - \tfrac{1}{2}\bar{\psi}(i\not{D}-m)\psi$$
$$+ \tfrac{1}{2}(F^\dagger F + G^\dagger G) + \tfrac{m}{2}(F^\dagger A + G^\dagger B + h.c.) + ie\left\{A^\dagger\bar{\lambda}\psi + B^\dagger\bar{\lambda}\gamma_5\psi + A^\dagger d B + h.c.\right\} \tag{13.17}$$

or, on eliminating the dummy fields d, F, G

$$\mathcal{L} = \text{const.}\ \mathrm{tr}_R\left\{-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\bar{\lambda}\not{D}\lambda\right\} - \tfrac{1}{2}(D_\mu A)^\dagger(D_\mu A) - \tfrac{1}{2}(D_\mu B)^\dagger(D_\mu B) - \tfrac{1}{2}\bar{\psi}(i\not{D}-m)\psi$$
$$- \tfrac{m^2}{2}(A^\dagger A + B^\dagger B) + ie\left\{A^\dagger\bar{\lambda}\psi + B^\dagger\bar{\lambda}\gamma_5\psi + h.c.\right\} + \tfrac{e^2}{2}(A^\dagger\tau B + B^\dagger\tau A)^2 \tag{13.18}$$

Note that if the matter-fields $\Psi_\pm$ belong to the adjoint representation of the local internal symmetry group then the quantities $X^\dagger Y Z$ reduce to $\mathrm{tr}_R\, X[Y,Z]$. In general however only the Yang-Mills field Φ itself must belong to the adjoint representation, and the matter-field can belong to any representation $\mathcal{D}$ containing the adjoint representation in the product $\mathcal{D}^* \otimes \mathcal{D}$.

Equations (13.17) (13.18) represent the simplest form of the supersymmetric Yang-Mills Lagrangian and the steps leading to them are summarized in the third column of table 2. Of course, (13.17) and (13.18) are no longer invariant with respect to the full gauge group (13.12) since we have chosen a particular gauge. In fact the only gauge invariance left in (13.1) and (13.1) is the conventional gauge invariance $B_\mu \to B_\mu + \partial_\mu B$ mentioned above. Comparing (13.18) with the conventional Yang-Mills Lagrangian (13.3) we see that it is not much more complicated. The only essential novelty is the appearance of the Yang-Mills fermion field λ which accompanies B_μ because it is in the same supermultiplet. Note that this λ-field itself has a conventional Yang-Mills kinetic term.

TABLE 2 (Sections 13 and 14)

Conventional Matter Fields ψ, ϕ Yang-Mills Fields B_μ	Super-Matter Fields $\Psi(A,B,\psi)$ Super Yang-Mills Field Φ	Super Yang-Mills Field in Special Gauge $\Phi(\ldots B_\mu, \lambda) = \Phi(B_\mu, \lambda)$
	YANG-MILLS INTERACTIONS	
$\bar{\psi}(i\not{\partial}-m)\psi$ $+$ $\frac{1}{2}(\partial_\mu\phi)^\dagger(\partial_\mu\phi) + \frac{m^2}{2}\phi^\dagger\phi$ $\Downarrow$ $\bar{\psi}(i\not{D}-m)\psi$ $+$ $\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$ $+$ $\frac{1}{2}(D_\mu\phi)^\dagger(D_\mu\phi) + \frac{m^2}{2}\phi^\dagger\phi$	$h\left\{-\frac{1}{8}\Psi_+^\dagger\Psi_+ + \frac{m}{2}\Psi_+^\dagger\Psi_- + h.c.\right\}$ $\Downarrow$ $h\left\{-\frac{1}{8}\Psi_+^\dagger e^{ie\Phi}\Psi_+ + \frac{m}{2}\Psi_+^\dagger\Psi_- + h.c.\right\}$ $+$ $h\left\{\frac{1}{8}\Phi_\mu^\dagger\Phi_\mu^+ + h.c.\right\}$	$\frac{1}{2}\bar{\psi}(i\not{\partial}-m)\psi$ $+\frac{1}{2}(\partial_\mu A)^\dagger(\partial_\mu A) + \frac{1}{2}(\partial_\mu B)^\dagger(\partial_\mu B) + \frac{m^2}{2}(A^\dagger A + B^\dagger B)$ $\Downarrow$ $\frac{1}{2}\bar{\psi}(i\not{D}-m)\psi$ $+\frac{1}{4}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}\bar{\lambda}\not{D}\lambda$ $+\frac{1}{2}(D_\mu A)^\dagger(D_\mu A) + \frac{1}{2}(D_\mu B)^\dagger(D_\mu B) + \frac{m^2}{2}(A^\dagger A + B^\dagger B)$ $-ie\left\{A^\dagger\bar{\lambda}\psi + B^\dagger\bar{\lambda}\gamma_5\psi + h.c.\right\}$ $+\frac{e^2}{2}(A^\dagger\tau B + B^\dagger\tau A)^2$
	SELF INTERACTIONS OF MATTER-FIELDS	
$g\bar{\psi}\phi\psi + V(\phi)$	$g\left\{\Psi_+^3 + \Psi_-^3\right\}_h$	$g\bar{\psi}(A + i\gamma_5 B)\psi$ $+ V(A, B, g, m, \lambda)$

The Yang-Mills interaction for superfields is compared with that for conventional fields. For clarity, the self-interaction of the matter field (g-coupling) is separated from the Yang-Mills interaction (e-coupling) and the indices in the self-interaction are suppressed.

14. Super Unified Gauge Theory.

Conventional unified gauge theory is obtained from Yang-Mills theory by[26] adding a scalar field ϕ with transformation properties $\phi' = S\phi S^{-1}$ to the Yang-Mills Lagrangian (13.1) to get

$$\mathcal{L} = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{2}(\mathcal{D}_\mu\phi)^\dagger(\mathcal{D}_\mu\phi) + g\bar{\psi}\phi\psi + V(\phi), \tag{14.1}$$

where

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \lambda^2\phi^4,$$

and then spontaneously breaking the symmetry by letting $\phi \to \mathring{\phi} + \phi$, where $\mathring{\phi}^2 = \frac{\mu^2}{2\lambda^2}$ so that $\langle\phi\rangle = 0$ at the potential minimum. One sees that in that case the vector meson fields B_μ, which hitherto are massless and lead to serious infra-red problems, acquire a mass-matrix of the form

$$-\frac{1}{2}(\mathcal{D}_\mu\mathring{\phi})^\dagger(\mathcal{D}_\mu\mathring{\phi}) = +\frac{g^2}{2}(B_\mu\mathring{\phi})^\dagger(B_\mu\mathring{\phi}). \tag{14.2}$$

The potential is chosen so that $\mathring{\phi}$ will produce masses for all but an abelian subset of B_μ fields. (The fermions also acquire a new mass-matrix $m - g\mathring{\phi}$ which in general does not commute with the group generators.)

From (13.18) we see that supersymmetry has the property that in the supersymmetric Yang-Mills Lagrangian the matter-fields ψ are automatically accompanied by scalar fields A and B corresponding to the scalar field ϕ which is inserted by hand in conventional case above. However, the multiplet $\Psi(A, B, \psi)$ is not automatically equipped with a potential corresponding to the matter-matter potential $V(\phi)$ above. In fact the only potential provided by the Yang-Mills theory itself comes from the terms

$$\frac{1}{2}\,\mathrm{tr}\,d^2 + ie\left(A^\dagger d B + B^\dagger d A\right) \rightarrow -\frac{1}{2} d_\alpha d_\alpha \ , \quad d_\alpha = -ie\left(A^\dagger \tau_\alpha B + B^\dagger \tau_\alpha A\right) \tag{14.3}$$

in (14.1) and as we shall now see, these make no contribution to spontaneous symmetry breaking (at least in the tree approximation). To see this, suppose that we have a spontaneous breaking $A \rightarrow A - \frac{m}{g} f$ coming from either (14.3) above or a combination of this (14.3) and some inserted potential. Then in (14.3) we have

$$d_\alpha \rightarrow d_\alpha + \frac{ime}{g}\left(f^\dagger \tau_\alpha B + B^\dagger \tau_\alpha f\right). \tag{14.4}$$

But since τ_α is a Yang-Mills group generator, we know from section that $\tau_\alpha f$ is a Goldstone direction. Hence the only contribution to (14.4) comes from the B particles which lie in Goldstone directions, and from Fig. 13 we see that it is precisely these particles that can be absorbed by the longitudinal part of the vector field, and hence vanish in the unitary gauge.[26] Thus (14.3) makes no contribution in the unitary, or physical, gauge.

Since, as we have just seen, the Yang-Mills theory itself does not provide a potential for spontaneous symmetry breaking, we must insert a matter-matter potential by hand, as in (14.1). But as well as being invariant under the Yang-Mills group (or some larger internal symmetry group containing the Yang-Mills group) such a potential must also be supersymmetric. We have seen in previous sections that the form of a supersymmetric potential is pre-determined and that it allows only one overall coupling constant g to describe the Yukawa, ϕ^3 and ϕ^4 coupling. Thus in the supersymmetric case the potential $V(\phi)$ cannot be chosen at will to produce any required mass-matrix (14.2) for the vector fields. The question therefore arises: does the predetermined supersymmetric potential produce a unified gauge theory? In other words, does the supersymmetric matter-matter potential allow spontaneous symmetry breaking, and if so, is the breaking such that all but an abelian subset of the vector fields acquire masses through the mass formula (14.2)?

We have seen in section twelve that a matter-matter supersymmetric potential does indeed allow spontaneous symmetry breaking. We shall now show, by considering the two classes of groups used in that section, that, whether the breaking is such as to provide all but an abelian set of the vector fields with masses, depends on the choice of group, the representation, and the number of matter fields used.

SU(N): The solutions of the fixed point equation (11.5) are such that the little group always contains an SU(2) group as subgroup. From (14.2) we see that the vector fields corresponding to this SU(2) subgroup do not acquire masses. Hence we cannot obtain a unified gauge theory for SU(N) using only one matter field.

With more than one matter-field, however, the situation changes. For example, if for SU(3) we take two matter-fields and allow them to break along different directions (orthogonal to I-spin and U-spin, say) as in Fig. 14, then

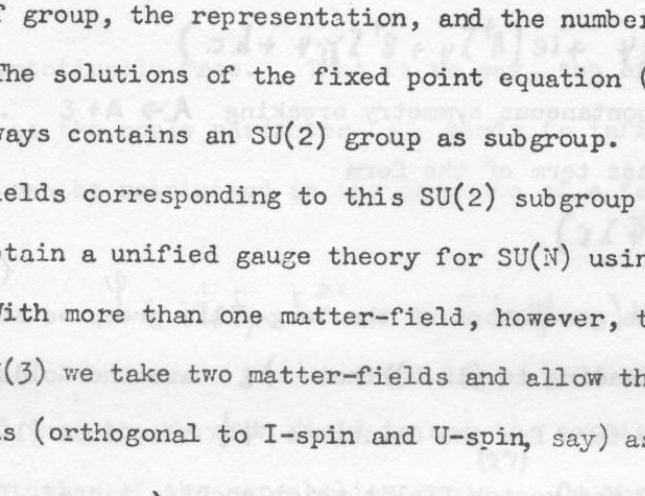


Fig. 14

from (14.2) we have

$$-\frac{1}{2}(D_\mu A)^\dagger (D_\mu A) \to -\frac{e^2}{2}(B_\mu A)^\dagger (B_\mu A) = -\frac{e^2}{2}\sum_{i=1}^{2}\left[B_\mu, A_i\right]^2 \Rightarrow -\frac{e^2}{2}\sum_{i=1}^{2}\left[B_\mu, \frac{m_i}{g_i} f_i\right]^2$$

$$= -\frac{e^2}{2}\left\{\left(\frac{m}{2g}\right)^2\left[B_\mu^2(4)+B_\mu^2(5)\right]+\left(\frac{\mu}{2f}\right)^2\left[B_\mu^2(1)+B_\mu^2(2)\right]+\left[\left(\frac{m}{2g}\right)^2+\left(\frac{\mu}{2f}\right)^2\right]\left[B_\mu^2(6)+B_\mu^2(7)\right]\right\}$$

from which we see that all but the abelian set $B_\mu(0)$, $B_\mu(3)$ of Yang-Mills fields acquire masses. This result is true, of course, only in the tree approximation and it is possible that the two directions will no longer remain different when we go to higher orders.

GxG: Taking the diagonal solution ($\kappa\delta_{i\alpha}$) of section 12 in the mass-formula (14.2) and recalling that the Y-M group is only one of the groups G in GxG we obtain in this case

$$-\tfrac{1}{2}(\partial_\mu A)^\dagger(\partial_\mu A) \Rightarrow -\frac{e^2}{2}(B_\mu A)^\dagger(B_\mu A) = -\frac{e^2}{2}(B_\mu A)_{i\alpha}(B_\mu A)_{i\alpha}$$

$$\Rightarrow -\frac{e^2}{2}\kappa^2\left(\epsilon_{i\alpha\beta}B^\beta_\mu\right)\left(\epsilon_{i\alpha\gamma}B^\gamma_\mu\right) = -\frac{e^2\kappa^2}{2}\,2B^\beta_\mu B^\beta_\mu = -e^2\kappa^2 B_\mu^2 . \tag{14.5}$$

Thus <u>all</u> the vector mesons acquire a mass, indeed the same mass. These results establish that, in principle at any rate, a superunified gauge theory is possible.

Finally we consider what happens to the fermion part

$$\mathcal{L}_F = -\tfrac{i}{2}\bar{\lambda}\not{\partial}\lambda - \tfrac{i}{2}\bar{\psi}(\not{\partial}+im)\psi + ie\left(A^\dagger\bar{\lambda}\psi + B^\dagger\bar{\lambda}\gamma_5\psi + h.c.\right) \tag{14.6}$$

of the Lagrangian (13.17) under spontaneous symmetry breaking $A \to A+\epsilon$. From (14.6) we see that we pick up a mass term of the form

$$ie\left(\epsilon^\dagger\bar{\lambda}\psi + \bar{\psi}\lambda\epsilon\right). \tag{14.7}$$

Since $\lambda = \lambda_\alpha\tau_\alpha$ where τ_α are the generators of the Yang-Mills group we see (by an argument similiar to that leading to (14.4)) that $\lambda\epsilon$ are the Goldstone directions. Hence (14.7) is a mass-term for the Goldstone ψ's and, from (14.2) with $\phi = \epsilon$, those λ's which partner the vector fields which acquire masses. There are the same number, n say, of each of these fields since each $\lambda^\alpha\tau_\alpha$ projects out one linear combination of ψ's , and hence for these fields there is a free Lagrangian

$$\mathcal{L}_F'^{(0)} = -\tfrac{i}{2}\bar{\lambda}_A\not{\partial}\lambda_A - \tfrac{i}{2}\bar{\psi}'_A\not{\partial}\psi'_A - \bar{\lambda}_A m_{AB}\psi_B, \qquad A,B=1\ldots n \tag{14.8}$$

where the mass-matrix $m_{AB} = \tau^A_{Bi}\epsilon_i$ is the same as the mass-matrix (14.2) for the vector fields. By introducing the <u>complex</u> spinor fields

$$\phi = \tfrac{1}{2}(u+iv) = \tfrac{1}{2}(\lambda+\psi) + \tfrac{i}{2}(i\gamma_5)(\lambda-\psi) \tag{14.9}$$

this free Lagrangian reduces to the form

$$\mathcal{L}_F'^{(0)} = -i\phi_A^\dagger\not{\partial}\phi_A - \phi_A^\dagger m_{AB}\phi_B \tag{14.10}$$

Equation (14.10) shows that there is a (supersymmetric) Higg's mechanism at work for the fermion partners λ of the massive vector fields. The latter acquire a mass not by absorbing the Goldstone fermions but by combining with them to form complex fermion fields. An interesting feature of this mechanism is that the full Lagrangian is then invariant under the gauge transformation $\phi_A \to e^{i\alpha}\phi_A$, so that the complex spinors ϕ_A, $A=1\ldots n$ thereby acquire also a conserved fermion number.

15. A Note on Asymptotic Freedom.

It is now known that although renormalizable abelian theories are not asymptotically free,[27] the simplest non-abelian Yang-Mills theory, namely the self-interacting theory,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ie\left[B_\mu, B_\nu\right] \tag{15.1}$$

is asymptotically free. That is to say, the effective coupling constant $e(\lambda)$ tends to zero as the scale parameter λ tends to infinity. Furthermore, the asymptotic freedom can be maintained in the presence of a fermion matter field

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\not{D} - m)\,\psi\,, \tag{15.2}$$

provided that the fermion field is not too exotic, i.e. does not belong to too large a representation of the non-abelian group.[27] On the other hand, if scalar matter-fields are also introduced (with a view to providing the vector-fields with masses by spontaneous symmetry breaking) then the asymptotic freedom is difficult to maintain without using exotic fermions.[27] The reason is that for the scalar field, the renormalizability requires the presence of a ϕ^4-interaction and this interaction introduces a second, independent, coupling constant $g(\lambda)$ which must also tend to zero as λ tends to infinity.

From (13.18) we see, however, that if a conventional Yang-Mills field is part of a Yang-Mills superfield, then even though there is only one independent coupling constant e, the Yang-Mills field is automatically in interaction with both fermion and boson fields, and that the latter already have a ϕ^4-type coupling (with constant proportional to e^2). If we assume that the interaction is renormalizable, then the renormalization group equations apply,[27][28] and it is interesting to ask whether in this case, with only one coupling constant involved, the theory is asymptotically free. This question can be answered by applying generally the Gross-Wilczek-Politzer formula[4][10] for the effective coupling constant $e(\lambda)$, namely, $\frac{d \log e(\lambda)}{d\lambda} = \beta(e(\lambda))$, where

$$\beta(e) = \frac{-e^2}{6.\,16\pi^2} C_2(G)\left\{22 - (4f_R + b_R)W(R)\right\} + O(e^4) . \tag{15.3}$$

Here f_R and b_R are the numbers of fermions and bosons belonging to the representation R of the Yang-Mills group, and $W(R)$ is a representation constant given by

$$W(R) = \varepsilon\left[tr_R\, C_2(R) \,/\, tr_R\, C_2(G) \right] \tag{15.4}$$

where $C_2(R)$, $G_2(G)$ are the Casimir operators for the adjoint and R representations, and $\varepsilon = 1, 2$ for self-conjugate and complex representations respectively. The criterion for asymptotic freedom is, of course, $\beta(e) < 0$.

Let us now consider the supersymmetric case, and suppose that we have one Yang-Mills superfield Φ interacting with n_R chiral scalar matter fields Ψ belonging to each representation R of the Yang-Mills group. Then, since the Yang-Mills superfield itself contains one fermion field, and each matter-field contains one fermion and two scalar fields, the formula (15.3) reduces in this case to (4)(10)

$$\beta(e) = -\frac{e^2}{16\pi^2} C_2(G)\left\{3 - n_R W(R)\right\} + O(e^4) \tag{15.5}$$

where $W(R)$ is given in (15.4). From this equation we see that if the representations R of the matter-fields are not too large relative to the adjoint representation, the Yang-Mills field can interact with a number of them without losing its asymptotic freedom.

This positive result for supersymmetric Yang-Mills fields may, perhaps, be understood by recalling that supersymmetry forces bosons to behave like fermions, and, as mentioned above, fermions do not disturb the asymptotic freedom provided that they are not too exotic. The positive result also shows that a theory which is not asymptotically free when all the coupling constants are independent, may become asymptotically free when the coupling constants are subjected to special constraints.

Finally the positive result for supersymmetric Yang-Mills fields raises the question as to whether it might be possible to construct a unified gauge theory which is asymptotically free, that is, to construct a theory which is both infra-red convergent and asymptotically free.

To consider this question let us take as example the unified gauge theories of the last section. These theories are infra-red convergent by definition, so the question is whether they are asymptotically free. From (15.5) we see that if we neglect the matter-matter interaction the condition for asymptotic freedom is

$$n_R W(R) < 3 . \tag{15.6}$$

If we now consider the matter fields $\Psi_{i\alpha}$ belonging to the adjoint $\otimes$ adjoint representation of GxG, where the Yang-Mills field belongs to the adjoint representation of G, we see that we have

$$W(R) = 1 \qquad \text{and} \qquad n_R = \text{order of G} . \tag{15.7}$$

Hence in this case (15.6) cannot be satisfied. The best we can do is take G = SU(2), in which case we have $n_R W(R) = 3$ and $\beta(e) = 0$ to order e^2 (10) (even then, it actually turns out that (21) $\beta(e) > 0$ in order e^4). On the other hand, if we let the Yang-Mills belong to the adjoint representation of SU(3), and take two matter fields also belonging to the adjoint representation of SU(3) and breaking along different directions as described in the last section, then we clearly have

$$W(R) = 1 \qquad \text{and} \qquad n_R = 2 \tag{15.8}$$

and so condition (15.6) is satisfied. (Even if the two matter-fields interact, it can be shown that, except for some special ratios of the coupling constants, the breaking is still along two independent directions.)

Unfortunately, the satisfaction of (15.6) is not a sufficient condition to guarantee asymptotic freedom. The reason is that, as discussed in the last section, in order to obtain the spontaneous symmetry breaking that produces the masses for the Yang-Mills fields, we have to introduce a WZ matter-matter interaction. This interaction introduces at least one more coupling constant g into the Lagrangian (at least two more in the SU(3) case above). These new WZ couplings are not asymptotically free in the absence of the Yang-Mills interaction, $\beta_i(g) > 0$ and the question, therefore, is whether the Yang-Mills interaction can make the matter-matter interaction asymptotically free, and remain asymptotically free itself in the presence of such an interaction.

In general, all the questions raised above may be summarized as follows: The matter-matter interaction has spontaneous breaking directions ε_a given by formulae such as

$$d_{abc}\,\varepsilon_b\,\varepsilon_c = \varepsilon_a \tag{15.9}$$

of section twelve. Is it now possible to choose a Yang-Mills group and assign the matter-fields to representations of it such that the following three conditions are satisfied simultaneously?

(i) The spontaneous breaking (15.9) provides masses for all but an abelian subset of the Yang-Mills fields according to the formula of section 14, namely,

$$M^2(B) = \tfrac{1}{2}\,(B_\mu\varepsilon)^\dagger\,(B_\mu\varepsilon)\,.$$

(ii) The Yang-Mills interaction is asymptotically free

$$\beta(e) \simeq a e^2 < 0\,.$$

(iii) The matter-matter interaction is asymptotically free

$$\beta_i(g) \simeq a_i e^2 + b_{ijk}\,g_j g_k < 0\,.$$

Note that because of the form of the Lagrangian (13.18) there is no g^2 correction to $\beta(e)$. Thus, although $\beta(e)$ is affected by the presence (Yang-Mills interaction) of the matter-fields through (15.5), it is not affected by the self-interaction of these fields.

Since the sign of $\beta_i(g)$, like that of $\beta(e)$, depends only on the group, the representations and the number of matter-fields, the question as to whether we can simultaneously satisfy (i), (ii) and (iii) is a purely algebraic question. Since $b_{ijk} > 0$, as discussed above, we see that such a choice will be possible in any case only for $a_i < 0$, and for certain sectors in $e - g_i$-space.*

---

* See note on this question added in proof (p.76).

References

(1) P. Ramond, Phys. Rev. D3, 2415 (1971)

(2) J.-L. Gervais, B. Sakita, Nucl. Phys. B34, 633 (1971)

A. Neveu, J. Schwarz, Nucl. Phys. B31, 86 (1971)

Y. Iwasaki, K. Kikkawa, Phys. Rev. D8, 440 (1973)

(3) J. Wess, B. Zumino, Nucl. Phys. B70, 39 (1974); Phys. Lett. 51B, 239 (1974)

(4) B. Zumino, Proc. Internat. Conf. High Energy Physics, London 1974, I-254.

(5) J. Wess, Bonn Summer School Lecture Notes (1974)

(6) A. Salam, J. Strathdee, Nucl. Phys. B76, 477 (1974)

(7) P. Roman, Theory of Elementary Particles (North-Holland 1960)

(8) A particle representation is given by A. Salam, J. Strathdee, Nucl. Phys. B80, 499 (1974)

(9) A. Salam, J. Strathdee, Review, Trieste Preprint IC/74/126 Nucl. Phys.(in press).

(10) A. Salam, J. Strathdee, Review, Trieste Preprint IC/74/42 Phys. Rev.(in press).

(11) J. Wess, B. Zumino, Phys. Lett. 49B, 52 (1974)

(12) J. Iliopoulos, B. Zumino, Nucl. Phys. 76B, 310 (1974)

(13) M.-S. Tsao, Phys. Lett. 53B, 381 (1974)

(14) A. Salam, J. Strathdee, Phys. Lett. 49B, 465 (1974)

(15) P. Fayet, J. Iliopoulos, Phys. Lett. 51B, 461 (1974)

(16) G. Hegerfeldt, J. Henning, Review, Fort. Phys. 16, 491 (1967); 17, 463 (1969)

(17) R. Firth, J. Jenkins, Durham Univ. Preprint (1974)

(18) A. Salam, J. Strathdee, Nucl. Phys. B80, 499 (1974; B84, 127 (1975)

P. Dondi, M. Sohnius, Nucl. Phys. B81, 317 (1974)

(19) R. Haag, J. Lopuszánski, M. Sohnius, Karlsruhe Univ. Preprint (1974)

(20) R. Brout, Nuovo Cim. 47A, 932 (1967)

L. Michel, L. Radicati, Proc. 5th Coral Gables Conf. (Benjamin, New York, 1968), 19

W. Cabbibo, Proc. Nobel Symposium (Almqvist and Wiksell, Stockholm 1968), 227

A. Pais, Proc. Nobel Symposium (Almqvist and Wiksell, Stockholm 1968), 215; and references therein.

(21) L. Michel, L. Radicati, Ann. Inst. H. Poincaré 18A, 185 (1973),
L. Michel, L. Radicati, Ann. Phys. 66, 758 (1971),
F. Pegoraro, J. Subba Rao, Nucl. Phys. B44, 221 (1972), and references therein

(22) G. Goldstone, Nuovo Cim. 19, 154 (1961)

(23) G. Racah, Rend dei Lincei 8, 108 (1950)

(24) C. N. Yang, R. Mills, Phys. Rev. 96, 191 (1954)

(25) J. Wess, B. Zumino (QED) Nucl. Phys. B78, 1 (1974)
S. Ferrara, B. Zumino (Y-M) Nucl. Phys. B79, 413 (1974)
S. Ferrara, E. Remiddi (QED) Phys. Lett. 53B, 347 (1974)

(26) S. Weinberg, Review, Proc. Aix-en-Provence Conference, J. Phys. 34, Coll. 1, 45 (1973)

(27) R. Crewther, Review, Proc. Aix Conference, J. Phys. 34, Coll. 1, 111 (1973)

(28) S. Ferrara, J. Iliopoulos, B. Zumino, Nucl. Phys. B77, 413 (1974)

(29) A. Salam, private communication

(30) W. Pauli, Ann. Inst. H. Poincaré 6, 109 (1936)

(31) J. Nilsson, D. Tchrakian, Göteborg Preprint (1974)

(32) U. Niederer, L. O'Raifeartaigh, Review, Fort. Phys. 22, 131 (1974)

(33) D. Gross, F. Wilczek, Phys. Rev. 8D 3633 (1973).

Appendix A. Representation-free Majorana Spinors.

Let γ_μ where $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ (30) be any set of Dirac matrices. Then $\gamma_\mu^\dagger$ and $-\tilde{\gamma}_\mu$ are also Dirac matrices and so there exist unitary matrices η and C such that

$$\gamma_\mu^\dagger = \eta \gamma_\mu \eta^{-1}, \qquad \tilde{\gamma}_\mu = -C\gamma_\mu C^{-1}. \tag{A1}$$

(The matrix C is the charge conjugation matrix, and the negative sign is taken because of the association of γ_μ with the electric current. A positive sign could be obtained by using $C\gamma_5$.) Eq. (A1) corresponds to the fact that the Dirac representation is pseudo-unitary and pseudo-orthogonal

$$S^\dagger(\Lambda)\eta S(\Lambda) = \eta, \qquad \tilde{S}(\Lambda) C S(\Lambda) = C. \tag{A2}$$

From (A2) one easily sees that $\eta^\dagger \eta^{-1}$ and $\tilde{C} C^{-1}$ are multiples of the identity. The phase of η can then be chosen so that $\eta^\dagger = \eta$ $(\Rightarrow \eta^2 = 1)$ and then the two sets of matrix covariants

$$\left(\eta \quad \eta\gamma_5 \quad \eta\gamma_\mu\right), \qquad \left(\eta\gamma_5\gamma_\mu, \quad \eta\sigma_{\mu\nu}\right), \tag{A3}$$

are hermitian and anti-hermitian respectively. In the case of C one sees at once that $\tilde{C} = \varepsilon C$ where $\varepsilon = \pm 1$. For $\varepsilon = -1$ the two sets of matrix covariants

$$\left(C, C\gamma_5, C\gamma_5\gamma_\mu\right), \qquad \left(C\gamma_\mu, C\sigma_{\mu\nu}\right), \tag{A4}$$

are anti-symmetric and symmetric respectively, while for $\varepsilon = +1$ the converse would be true. However, a four-dimensional space permits only six independent anti-symmetric matrices and hence $\varepsilon = -1$.

From (A2) we see at once that the Dirac representation is self-conjugate

$$S^*(\Lambda) = \left(\tilde{\eta}^{-1} C\right) S(\Lambda) \left(\tilde{\eta}^{-1} C\right)^{-1}. \tag{A5}$$

From the self-conjugacy and $\tilde{C} = -C$ it follows that ψ^* and $\tilde{\eta}^{-1}\tilde{C}\psi$ have the same Lorentz transformation properties and hence that it is consistent with Lorentz covariance to identify them. The general Majorana spinors are those spinors for which this identification is made. That is, the general Majorana spinors are defined to be those spinors which satisfy the Lorentz covariant reality condition

$$\psi^* = (\tilde{\eta}^{-1}\tilde{C})\psi. \tag{A6}$$

This condition can also be written in the form

$$\bar{\psi} \equiv \psi^\dagger \eta = \tilde{\psi} C, \tag{A7}$$

so that the Majorana spinors can also be thought of as those spinors for which the two sets of sixteen covariants $\psi^\dagger \eta \Gamma_A \psi$ and $\tilde{\psi} C \Gamma_A \psi$ coincide, where $\Gamma_A = 1, \gamma_\mu, \gamma_5 \gamma_5\gamma_\mu, \sigma_{\mu\nu}$. Note from (A3) and (A4) that the only hermitian covariants that one can form with anti-commuting Majorana spinors are

$$\theta\cdot\theta \quad , \quad \theta\cdot\gamma_5\theta \quad , \quad i\theta\cdot\gamma_5\gamma_\mu\theta, \tag{A8}$$

where $\theta\cdot\psi = \bar{\theta}\psi = \tilde{\theta}C\psi$ as in eq. (1.2).

If we now let $\vec{\sigma}$ be the Pauli matrices, and for the metric $g_{\mu\mu} = -2$ use the special (Majorana, or pure imaginary) realization

$$\gamma_0 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, \quad \gamma_1 = i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \quad \gamma_3 = i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{A9}$$

of the Dirac matrices, we see that $C = \eta = \gamma_0$, and the above formalism simplifies to $S^*(\Lambda) = S(\Lambda)$ and $\psi^* = \psi$, as anticipated in section one. We also have

$$\gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma_5^\dagger = \gamma_5, \quad \gamma_5^* = -\gamma_5. \tag{A10}$$

Appendix B. Weight Diagrams for Higher Spin Superfields.

Higher spin superfields have the transformation laws

$$\left(U(a,\Lambda)\,\Phi\right)(x,\theta) = \mathcal{D}(\Lambda)\,\Phi\left(\Lambda^{-1}(x-a), S^{-1}\theta\right), \quad \left(U(\varepsilon)\Phi\right)(x_\mu,\theta) = \Phi\left(x_\mu + \tfrac{i}{2}\bar{\theta}\gamma_\mu\varepsilon, \theta+\varepsilon\right), \tag{B1}$$

that is, the same laws as the scalar fields except for the transformation $\mathcal{D}(\Lambda)$ of the external Lorentz indices, which itself is the same as for conventional higher spin fields.[31] The laws (B1) imply that the momentum commutes with the supersymmetry transformations (just as in the scalar case, in which the commutation is shown explicitly in (1.7)). It follows that the external components of $\Phi(x,\theta)$ can be made to correspond to a single spin by means of a conventional wave-equation, e.g. $\partial_\mu \Phi_\mu(x\theta) = 0$.[32] There are, of course, many different choices of $\mathcal{D}(\Lambda)$ and corresponding wave-equation which one can use to describe a single spin, and for supersymmetry it turns out that one very convenient choice is to let $\mathcal{D}(\Lambda)$ be the Joos-Weinberg[32] representation $\mathcal{D}^j \equiv \mathcal{D}^{(j0)} \oplus \mathcal{D}^{(0j)}$. Then the only wave-equation is the JW condition for definite parity. The reason that the JW representation is so convenient is that when multiplied with spinors and four-vectors it has a very simple Clebsch-Gordon decomposition, namely,

$$\mathcal{D}^j\,\mathcal{D}^{1/2} \equiv \left(\mathcal{D}^{(j0)} \oplus \mathcal{D}^{(0j)}\right)\left(\mathcal{D}^{(1/2\,0)} \oplus \mathcal{D}^{(0\,1/2)}\right) = \mathcal{D}^{j+1/2} \oplus \mathcal{D}^{j-1/2} \oplus \Delta^{j+1/2} \tag{B2}$$

and

$$\mathcal{D}^j\,\mathcal{D}^{(1/2,1/2)} \equiv \left(\mathcal{D}^{(j0)} \oplus \mathcal{D}^{(0j)}\right)\mathcal{D}^{(1/2,1/2)} = \Delta^{j+1} \oplus \Delta^j \tag{B3}$$

where

$$\Delta^j \equiv \mathcal{D}^{(j-1/2,\,1/2)} \oplus \mathcal{D}^{(1/2,\,j-1/2)} .$$

Accordingly if we make an expansion of $\Phi^j_m(x,\theta)$ corresponding to the expansion (1.2) of the scalar field we obtain

$$\Phi^j_m(x,\theta) = A^j_m(x) + \theta\cdot\Psi^j_m(x) + \tfrac{1}{4}\theta\cdot\left[F^j_m(x) + \gamma_5 G^j_m(x) + i\gamma_5 \not{B}^j_m(x)\right]\theta + \tfrac{1}{4}(\theta\cdot\theta)\,\theta\cdot\chi^j_m(x) + \tfrac{1}{32}(\theta\cdot\theta)^2 D^j_m(x) \quad \text{(B4)}$$

where the fields Ψ, χ and B_μ are now linear combinations of two and three fields

$$\Psi^j_{m\alpha} = \sum_s C^{j\,\frac{1}{2}\,s}_{m\,\alpha\,m+\alpha}\,\Psi^s_{m+\alpha}, \qquad B^j_{m\mu} = \sum_s \Gamma^{j\,1\,s}_{m\,\mu\,m+\mu}\,B^s_{m+\mu}, \quad \text{(B5)}$$

and similarly for χ, where the C's and Γ's are the C-G coefficients for the expansions (B2) and (B3) respectively.

The weight diagrams for such higher spin superfields are then clearly of the form

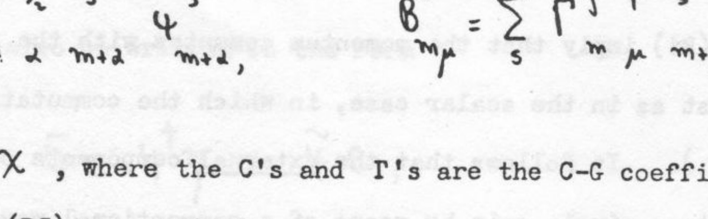


Fig. 16

where the arrowheads denote the highest and lowest spins in the Lorentz multiplets. Thus the central weights such as w are singly, trebly and quadruply degenerate for p = 0, 4, p = 1, 3 and p = 2 respectively (still suppressing the multiplicity <u>within</u> Lorentz multiplets).

By inspection of Fig. 15 we see that there are no Lorentz scalar weights (i.e. no coincidence of arrowheads pointing in opposite directions), and hence no candidates for Lagrangian densities, unless $j = 0$ or $j = \frac{1}{2}$. The case $j = 0$ is the scalar case already depicted in Fig. 1 and the possible candidates are the fields A, F, G and D. The diagram for $j = \frac{1}{2}$ is shown in Fig. 17,

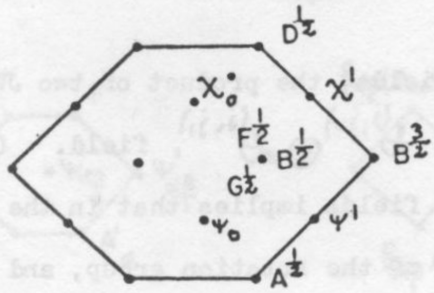


Fig. 17

and from this diagram we see that there are two candidates, χ_0 at $p = 3$ and ψ_0 at $p = 1$. Of these six candidates altogether, only D, the highest weight of the scalar multiplet, is <u>guaranteed</u> to be a supersymmetric Lagrangian density, i.e. to have a supersymmetric invariant space-time integral. The other candidates will be supersymmetric Lagrangian densities if, and only if, the higher weights in their multiplets vanish on integration.

To form the product of two superfields of higher spin, Φ^{j_1} and Φ^{j_2} say, we note that the product is a linear combination of higher spin superfields. To determine which superfields occur in the linear combination we then note that the higher spin symmetry $\mathcal{D}(\Lambda)$ commutes with the supersymmetry

$$[\mathcal{D}(\Lambda), G_\alpha] = 0, \tag{B6}$$

and that every state in a supermultiplet can be generated from the lowest weight states by repeated application of G_α. It follows that the decomposition of the superfields is completely determined by the decomposition of the lowest weight fields. But the lowest weight fields are conventional fields carrying the same external indices as the superfields. Hence the superfields decompose in exactly the same way as conventional fields.

The decomposition is not completely trivial, however, because

$$\left[\mathcal{D}^{(j_1 0)} \oplus \mathcal{D}^{(0 j_1)}\right] \otimes \left[\mathcal{D}^{(j_2 0)} \oplus \mathcal{D}^{(0 j_2)}\right] = \left\{\sum_{j=|j_1-j_2|}^{j_1+j_2} \mathcal{D}^{(j 0)} \oplus \mathcal{D}^{(0 j)}\right\} \oplus \left\{\mathcal{D}^{(j_1 j_2)} \oplus \mathcal{D}^{(j_2 j_1)}\right\}, \qquad (B7)$$

and hence, even for conventional fields, the product of two JW fields contains not only a sum of JW fields but also a $\mathcal{D}^{(j_1 j_2)} \oplus \mathcal{D}^{(j_2 j_1)}$ field. On the other hand, the JW parity condition for the component fields implies that in the rest-frame they describe single irreducible representations of the rotation group, and hence implies that the product contains only the representations $j_1 + j_2 \geq j \geq |j_1 - j_2|$ of the rotation group, each j occurring with multiplicity one. It follows that there exists a unitary transformation that eliminates the $\mathcal{D}^{(j_1 j_2)} \oplus \mathcal{D}^{(j_2 j_1)}$ field, and provides a JW condition for each of the irreducible $\mathcal{D}^{(j0)} \oplus \mathcal{D}^{(0j)}$ fields. <u>In this general sense</u> we may say that the product of two conventional JW fields A^{j_1} and A^{j_2} decomposes into the JW fields A^j where $j_1 + j_2 \geq j \geq |j_1 - j_2|$, and hence that the product of two JW superfields Φ^{j_1} and Φ^{j_2} decomposes into the JW superfields Φ^j where $j_1 + j_2 \geq j \geq |j_1 - j_2|$. For example,

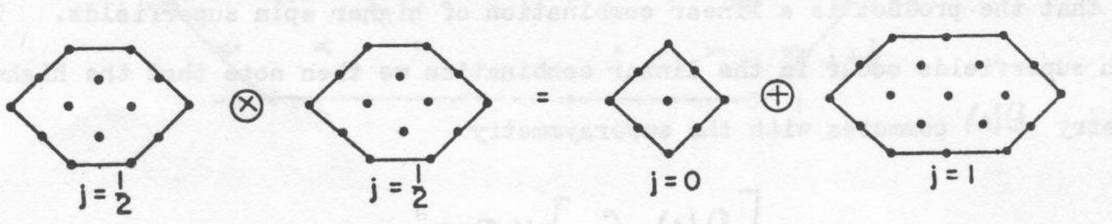


Fig. 18

Of course, since the higher weights are combinations of products of lower weights even in the scalar case as depicted in Fig. 2, the higher weights in the products are quite complicated. The complication is further increased by the fact that the C-G decomposition (B5) for the inner and outer indices must be re-arranged to take account of the unitary transformation which reproduces only JW representations in the product. We shall not discuss these complications in general here, but only give the full decomposition for the simplest possible case, namely the product of two spin $\frac{1}{2}$ chiral fields, for which we can use the representations $\mathcal{D}^{(j0)}$ to avoid the JW parity problem. For this case we have

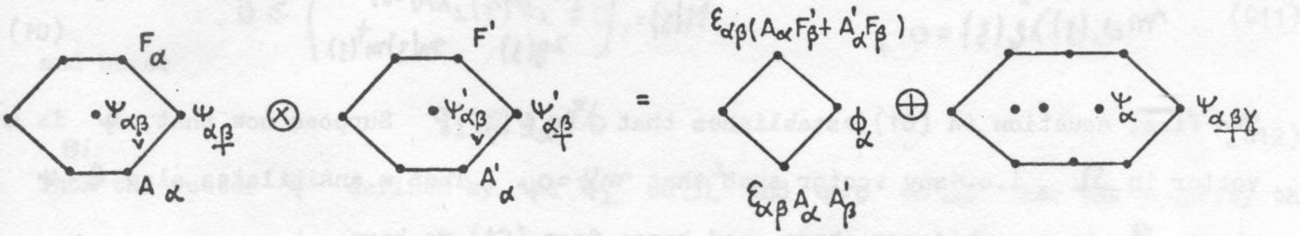


Fig. 19

where $\underline{\alpha\beta}$ and $\underset{\vee}{\alpha\beta}$ denote symmetrization and anti-symmetrization respectively,

$$\phi_\alpha = \varepsilon_{\beta\gamma}\, \psi_{\alpha\beta} A_\gamma$$

$$\psi_\alpha = \varepsilon_{\beta\gamma}\, \psi_{\beta\alpha} A_\gamma - \varepsilon_{\beta\gamma}\, \psi_{\beta\gamma} A_\alpha \quad (= \phi_\alpha^\perp), \tag{B8}$$

$\psi_{\alpha\beta\gamma}$ is the totally symmetric part of $\psi_{\alpha\beta} A_\gamma$, and

$$\Phi_\alpha(x\theta) = A_\alpha + \psi_{\alpha\beta}\,\theta^\beta + \frac{1}{4}(\theta\cdot\theta)\, F_\alpha + \cdots . \tag{B9}$$

The fields in (B8) are orthogonalized but not yet normalized.

Appendix C. Proof of Lemma of Section 9.

We wish to establish here the lemma of section 9. To establish part (a) we note from (9.7) and (9.8) that the necessary and sufficient conditions for a local minimum are

$$m_{ab}(\mathfrak{z})\lambda^*_b(\mathfrak{z}) = 0 \,, \quad M(\mathfrak{z}) = \begin{pmatrix} m^\dagger(\mathfrak{z})m(\mathfrak{z}) & 2g^\dagger(\mathfrak{z}) \\ 2g(\mathfrak{z}) & m(\mathfrak{z})m^\dagger(\mathfrak{z}) \end{pmatrix} \geq 0 \,. \tag{C1}$$

The first equation in (C1) establishes that $\lambda^*(\mathfrak{z}) \in \Omega$. Suppose now that ψ is any vector in Ω, i.e. any vector such that $m\psi = 0$. Then m annihilates also $e^{i\theta}\psi$, where θ is an arbitrary phase, and hence from (C1) we have

$$\mathrm{Re}\, e^{2i\theta}(\psi, g(\mathfrak{z})\psi) \geq 0 \tag{C2}$$

But for arbitrary θ this is not possible unless

$$\mathrm{Re}\,(\psi, g(\mathfrak{z})\psi) = 0 \,. \tag{C3}$$

But then

$$(\psi^* \ \psi)\Big(M(\mathfrak{z}) \Big)\begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = 0 \,, \tag{C4}$$

and hence, since $M(\mathfrak{z})$ is positive

$$M(\mathfrak{z})\begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = 0 \,. \tag{C5}$$

Then using $m\psi = 0$ again we obtain

$$\begin{pmatrix} 0 & g^\dagger(\mathfrak{z}) \\ g(\mathfrak{z}) & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi^* \end{pmatrix} = 0 \quad \Rightarrow \quad g(\mathfrak{z})\psi = 0 \,. \tag{C6}$$

Thus $g(\mathfrak{z})\Omega = 0$ as required. Finally the inequality (iii) in the lemma is just the inequality in (C1) restricted to $\Omega^\perp$.

To establish part (b) of the lemma we suppose that $\det(m_\perp - \omega g_\perp)$ is not independent of ω. Then the equation

$$\det(m_\perp - \omega g_\perp) = 0 \tag{C7}$$

is a polynomial equation for ω and has at least one solution $\omega = \mathring{\omega}$. Consider now the point $A_a = \mathfrak{z}'_a = \mathfrak{z}_a - \frac{1}{2}\mathring{\omega}\lambda^*_a$. From the corollary to the lemma of section 9 and the identity (9.6), it is easy to see that

$$m(\mathfrak{z}') = m(\mathfrak{z}) - \mathring{\omega}\, g(\mathfrak{z}) \tag{C8}$$

$$\lambda(\mathfrak{z}') = \lambda(\mathfrak{z}) \,. \tag{C9}$$

Equation (C7) for $\omega = \mathring{\omega}$ implies that there exists a vector $\psi_\perp$ such that

$$\left(m_\perp(\zeta) - \mathring{\omega}\, g_\perp(\zeta) \right) \psi_\perp = 0, \tag{C10}$$

and since $m_\perp(\zeta)$ is by definition non-singular,

$$m_\perp(\zeta)\psi_\perp \neq 0, \tag{C11}$$

and hence

$$g_\perp(\zeta)\psi_\perp \neq 0. \tag{C12}$$

Then the vector ψ defined by $\psi = \psi_\perp$ on $\Omega^\perp$ and $\psi = 0$ on Ω has the property that

$$\left(m(\zeta) - \mathring{\omega}\, g(\zeta) \right) \psi = 0, \qquad g(\zeta)\psi \neq 0. \tag{C13}$$

But from (C9) it follows in particular that

$$g(\zeta) = g(\zeta'). \tag{C14}$$

Hence from (C8) and (C14) it follows that

$$m(\zeta')\psi = 0, \qquad g(\zeta')\psi \neq 0. \tag{C15}$$

From part (a) of the lemma it then follows that the point ζ' cannot be a minimum of the potential. However, from (C9) we have in particular

$$V(\zeta) = V(\zeta'). \tag{C16}$$

Hence the point ζ cannot be an absolute minimum. Thus a necessary condition for an absolute minimum is that equation (C7) have no solutions for ω, and this can only happen if the determinant is independent of ω, which is the required result.

Appendix D. Intrinsic Parity.

We wish to motivate the definitions of parity and intrinsic parity given in section 9. For this purpose we first diagonalize the mass matrix m_{ab} by a gauge transformation S of the form (9.3), i.e. assume that $m_{ab} = m_a \delta_{ab}$, and note that there is no loss of generality in assuming that $m_a \geq 0$, since the phases of the m_a can be absorbed in S. Then, relative to positive m_a we define the parity operator P to be

$$P\, \Phi^a_\pm(x\theta) = \eta_a\, \Phi^a_\mp(x^P, i\gamma_0\theta), \tag{D1}$$

where η_a is the intrinsic parity. Note that η_a is intrinsic only relative to positive m_a, since a gauge transformation (9.3) of the form

$$\Psi_\pm \rightarrow \pm i \Psi_\pm \tag{D2}$$

simultaneously changes the signs of η_a and m_a. This corresponds to the situation for conventional fermion fields, for which the parity is intrinsic only relative to the mass, because a γ_5-transformation simultaneously changes the signs of η_a and m_a. Indeed for the conventional fields A, B, ψ contained in $\Psi_\pm$, the gauge-transformation (D2) is just

$$A \rightarrow -B, \qquad B \rightarrow A, \qquad \psi \rightarrow i\gamma_5 \psi, \tag{D3}$$

and $m_a \bar{\psi}_a \psi_a$ is the mass-term for the fermion field. Note that (D3) is just the chiral transformation C(0) of section 4. Note also that if η_a is the intrinsic parity of the superfield, the intrinsic parities of the fields A, ψ, B contained in it are η_a, $\pm i\eta_a$, $-\eta_a$ respectively.

With the definition (D1) of parity, a sufficient condition for parity conservation in the Lagrangian (9.2) is clearly

$$m_{ab} \geq 0, \qquad \lambda^* = \lambda_a \eta_a, \qquad g^*_{abc} = g_{abc}\eta_a\eta_b\eta_c . \tag{D4}$$

In practice, however, (D4) is not a convenient condition to use and hence for those fields for which η_a is negative, we make a gauge transformation of the form (D2). The definition (D1) of P is then replaced by

$$\left(P\Psi_\pm\right)(x\theta) = \Psi_\pm(x^P, i\gamma_0\theta), \tag{D5}$$

and a sufficient condition for parity conservation is simply

$$m^*_{ab} = m_{ab}, \qquad \lambda^*_a = \lambda_a, \qquad g^*_{abc} = g_{abc} \tag{D6}$$

Thus parity conservation reduces to the reality of the parameters. The price we pay for this simplification is that m_{ab} is no longer positive. In fact it is clear from the construction that the intrinsic parity is now given by

$$\eta_a = \operatorname{sgn} m_a, \qquad m_{ab} = m_a \delta_{ab} \tag{D7}$$

Equations (D5), (D6) and (D7) are those used in section 9. Note that the parity conservation condition in (D6) is only sufficient because degeneracies in m_{ab} and g_{abc} may allow some arbitrariness in the assignments.

Appendix E. Second-Order Corrections to Mass-Formulae.

In this appendix we compute the second order corrections to the mass-formulae of sections 9 and 10. We therefore consider a Lagrangian with

$$\mathcal{L}_m = -\tfrac{1}{2}(\mu^2)_{ab} A_a A_b - \tfrac{1}{2}(\sigma^2)_{ab} B_a B_b - \tfrac{1}{2} m_{ab} \bar{\psi}_a \psi_b \tag{E1}$$

and

$$\mathcal{L}_{int} = g_{abc} \bar{\psi}_a (A_b - \gamma_5 B_b)\psi_c + m_{ad} g_{dbc}\{A_a A_b A_c + 2A_b B_a B_c - A_a B_b B_c\} + g_{abc} g_{ade}\{(A_b A_c - B_b B_c)(A_d A_e - B_d B_e) + 4 A_b A_d B_c B_e\}. \tag{E2}$$

The interaction is the same as in the symmetric limit, but we must note that the mass used in the cubic term is the fermion mass in $\mathcal{L}_m$. The fact that $\mathcal{L}_{int}$ is the same as in the symmetric limit has two consequences. First in the super-symmetric limit $\mu = \sigma = m$ all the usual supersymmetric results hold - all mass and wave-function renormalizations are equal and the Yukawa vertex renormalization is finite. In particular, by considering only the two fermion self-mass diagrams, we can compute the symmetric limit mass and coupling constant renormalizations, and in particular the latter is found to be

$$\overset{(2)}{\Delta} g_{abc} = 3 g^{ad} g_{dbc}\, I \tag{E3}$$

where

$$g^{ab} = g_{acd}\, g_{bcd}\,, \qquad I = K\int \frac{d^4k}{(k^2-m^2)^2}\,, \quad K = \text{constant} \tag{E4}$$

the mass m being some average mass of m_{ab}, whose variation clearly changes I by only a finite amount.

The second consequence of $\mathcal{L}_{int}$ taking its symmetric limit value is that the spontaneous breaking affects only the masses in the Feynman propagators but not the Feynman vertices, and hence the one-loop graphs which are logarithmically divergent, will differ from their symmetric limits by only finite amounts (as in the case of the integral I above). Hence for renormalization purposes logarithmically divergent graphs may be regarded as unchanged, and the only corrections to the mass-formulae come from the graphs which are quadratically divergent (and cancel each other in the symmetric limit). There are only three such graphs in the 2nd order approximation, and one of them, the fermion loop contribution to the boson self-masses, actually remains unchanged because the fermion masses are not affected by the spontaneous breaking. The other two quadratically divergent graphs are easily seen from $\mathcal{L}_{int}$ in (E2) to be

$$
\begin{aligned}
&= 4 g_{aed}\, g_{bcd} (I^+ + I^-)_{ce} \\
&\pm 2 g_{abd}\, g_{dce} (I^+ - I^-)_{ce} ,
\end{aligned}
\qquad I^{\pm}_{ce} = \kappa \int \frac{d^4k}{(k^2 - m_{\pm}^2)_{ce}} . \tag{E5}
$$

If we denote by δ the difference between any quantity and its symmetric limit, and note that mass-differences vanish in the symmetric limit, we obtain from (E5)

$$
\overset{(2)}{\Delta}(m_+^2 + m_-^2 - 2m^2)_{ab} = \delta \overset{(2)}{\Delta}(\mu^2 + \sigma^2 - 2m^2) = 4 g_{aed}\, g_{bcd} (\mu^2 + \sigma^2 - 2m^2)_{ce}\, I , \tag{E6}
$$

$$
\overset{(2)}{\Delta}(\mu^2 - \sigma^2)_{ab} = \delta \overset{(2)}{\Delta}(\mu^2 - \sigma^2)_{ab} = 4 g_{abd}\, g_{cde} (\mu^2 - \sigma^2)_{ce}\, I . \tag{E7}
$$

Equations (E6) and (E7) show that the 2nd order corrections to the quantities $\mu^2 + \sigma^2 - 2m^2$ and $\mu^2 - \sigma^2$ are proportioned to these quantities themselves and hence that the mass-formulae

$$
\mu^2 + \sigma^2 = 2m^2 \qquad \text{or} \qquad \mu^2 = \sigma^2 \tag{E8}
$$

are maintained in the 2nd order approximation. Further, if the quantities $g_{aed}\, g_{bcd}$ and $g_{abd}\, g_{cde}$ are the same, as is certainly the case in section 10 where we have only one field, we can take an arbitrary combination of (E6) and (E7) and find that any mass formula

$$\alpha\mu^2 + \beta\sigma^2 + \gamma m^2 = 0, \qquad \alpha+\beta+\gamma = 0, \tag{E9}$$

is maintained in the one-loop approximation.

If the mass-difference comes about by spontaneous breaking and we have

$$(\mu^2-\sigma^2)_{ab} = 2 g_{abc}\lambda_c , \tag{E10}$$

then inserting this formula in (E7) we obtain

$$\overset{(2)}{\Delta}(\mu^2-\sigma^2)_{ab} = 8\, g_{abc}\, g^{cd}\, \lambda_d\, I .$$

Thus (E10) will be maintained if there exists a renormalization of λ such that

$$8 g_{abc}\, g^{cd} \lambda_d I = \overset{(2)}{\Delta}(2 g_{abc}\lambda_c) \tag{E11}$$

(and such that the only other condition on λ, namely $m_{ab}\lambda_c = 0$, is not violated). Using (E3) one sees at once that such a renormalization is given by

$$\overset{(2)}{\Delta}\lambda_a = g^{ab}\lambda_b I . \tag{E12}$$

With this renormalization the mass-formula (E10) (and the condition $m_{ab}\lambda_b = 0$) is maintained (up to finite corrections) after the one-loop correction.

Note (added in proof) on question raised at end of Section 15.

The answer to the question raised at the end of Section 15 (p.60) is generally negative for conventional fields[33], but we now show that the situation for supersymmetry is much more hopeful. The main reason for this is the following: For conventional theories[33], with, say, one matter-matter coupling constant g, the analogue of (iii) is

$$\beta(g) = Ag^2 + Be^2 + C\frac{e^4}{g^2}, \tag{15.10}$$

where the last term arises because the Yang-Mills interaction generates a quadrilinear matter-matter interaction in fourth order. For (15.10), the conditions for asymptotic freedom turn out to be[33]

$$B < a < 0, \qquad A > 0, \qquad (B-a)^2 < 4AC. \tag{15.11}$$

For supersymmetry, however, $C=0$ because the quadrilinear interaction coupling constant g^2 is the square of the Yukawa interaction coupling constant g, for which $C=0$ in both the conventional and supersymmetric theories. But for $C=0$ an analysis similar to that of Ref. (33) shows that in this case the conditions for asymptotic freedom are exactly the same as (15.11), except that condition $(B-a)^2 < 4AC$ is missing. But $(B-a)^2 < 4AC$ constitutes one of the main obstacles[33] to asymptotic freedom in the conventional case. Thus the form (iii) on p. 60 already removes one of the main obstacles.

A further bonus from supersymmetry is that b_{ijk} in (iii) is automatically of the form

$$b_{ijk}\, g_j g_k = \sum_j b_{ij}\, g_j^2, \qquad b_{ij} > 0, \tag{15.12}$$

and is even independent of the index i if the matter-matter interaction is symmetric in all fields. The reason is that since in supersymmetry the vertex renormalization is zero, the only contribution to b_{ijk} comes from the wave-function renormalizations of the matter-fields, which are all positive (and are equal if we have the permutation symmetry). From (15.12) it follows that the generalization of the condition $A>0$ in (15.11) is automatically satisfied. Thus the only question for supersymmetry is whether the analogue of B in (15.10) is negative and less than a, that is, whether in (iii) $a_i < a < 0$.

The value of a_i has been evaluated for me by Dr. Seán Browne, and it turns out that for a wide class of matter-matter interactions the constants a_i are indeed negative, and less than a , and that they are actually independent of the form of the matter-matter interaction. More precisely, one finds that, if the Yang-Mills and matter-matter interactions τ^{α}_{ab} and d^{a}_{bc} are such that

$$\tau^{\alpha}_{ab}\, d^{c}_{bd}\, \tau^{\alpha}_{de} = -\frac{K(R)}{2}\, d^{c}_{ae} \tag{15.13}$$

(for which it is sufficient that the matter-matter d-coupling be unique), then a_i is negative, and

$$\frac{a_i}{a} = \frac{6}{3-n(R)W(R)} \cdot \frac{K(R)}{C_2(G)} . \tag{15.14}$$

Thus the condition is satisfied provided $K(R) \geqslant C_2(G)/2$.

For the adjoint representations of $SU(N)$, $K = C_2(G)$, so the condition is indeed satisfied.

The net result of all this discussion is that, in contrast to the conventional case, for supersymmetry the introduction of a matter-matter interaction does not seem to generate any new difficulties. So the only problem is to satisfy the old conditions (i) and (ii). We have seen that, in principle, this can be done by using two multiplets breaking in different directions. The only question then remaining is whether such a configuration is stable, i.e. whether these solutions might collapse to a single (possibly parity violating) direction when we go beyond the tree approximation. There is, however, no particular reason to believe that a collapse will take place; so perhaps models such as this $SU(3)$ model are stable. If so (and if (15.14) is correct), supersymmetry provides theories which are infra-red convergent and asymptotically free in a reasonably natural way.